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by

H.F. Tiersten and M. Jahanmir

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ABSTRACT

The differential equations and boundary conditions describing the behavior of a finitely deformable, heat-conducting composite material are derived by means of a systematic application of the laws of continuum mechanics to a well-defined macroscopic model consisting of interpenetrating solid continua. Each continuum represents one identifiable constituent of the N-constituent composite. The influence of viscous dissipation is included in the general treatment. Although the motion of the combined composite continuum may be arbitrarily large, the relative displacement of the individual constituents is required to be infinitesimal in order that the composite not rupture. The linear version of the equations in the absence of heat conduction and viscosity is exhibited in detail for the case of the two-constituent composite. The linear equations are written for both the isotropic and transversely isotropic material symmetries. Plane wave solutions in the isotropic case reveal the existence of high-frequency (optical type) branches as well as the ordinary low-frequency (acoustic type) branches, and all waves are dispersive. For the linear isotropic equations both static and dynamic potential representations are obtained, each of which is shown to be complete. The solutions for both the concentrated ordinary body force and relative body force are obtained from the static potential representation.

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1. Introduction

A composite material is composed of a number of distinct identifiable constituents with different physical properties, which are combined to form a single solid. In particular, fiber-reinforced composites consist of fiber reinforcement imbedded in a matrix. In order to obtain certain types of information, e.g., the actual bond stress between the reinforcement and matrix, the individual elements of the composite must be treated as separate entities[1,2]. Nevertheless, it must be remembered that it would be virtually impossible to consider an external boundary of a composite while considering the individual elements in completely separate detail. On the other hand, in order to obtain certain other types of information, e.g., the effective elastic constants of the single composite material, some sort of single continuum model can readily be employed[3,4]. However, there are numerous other situations, e.g., the wave velocity dispersion induced by the fiber reinforcement in the matrix and possible resonances at which the reinforcement might separate from the matrix, for which neither of the aforementioned approaches can adequately or conveniently account but a different model somewhere between the two can account. Since composite materials composed of reinforcement spaced uniformly densely in the matrix can be modeled as interpenetrating solid continua and such a model can conveniently account for much of the physical phenomena that neither of the aforementioned approaches can, we employ this model in obtaining a description of composite materials. At this point we note that for the interpenetrating solid continuum model to be valid a characteristic length, such as a wavelength, must be large compared with, say, the spacing of the fiber reinforcement in much the same way that the wavelength of an elastic wave must be large compared with a lattice spacing for the ordinary elastic continuum description to be valid.

The interpenetrating solid continuum model, which is closely related to the model of fluid mixtures[5], has been employed in the description of a variety of physical phenomena such as, e.g., certain types of magnetoelastic interaction[6], electroelastic interaction[7] and the interaction of the electromagnetic field with deformable insulators[8]. In this latter case in order to consider ionic polarization resonances, the model consisted of two interpenetrating continua in which the motion of the center of mass of the two continua was finite but the relative motion of each of the continua with respect to the center of mass was infinitesimal. It was felt that in order for the description to be physically meaningful, the relative displacement of the two continua had to be infinitesimal, or else the solid would rupture. The application of this model to the description of material composites should be obvious. Indeed, the idea of employing interpenetrating continua as a model of composite materials has already been introduced by Bedford and Stern[9,10]. However, there are a number of fundamental differences between the approach of Bedford and Stern and that employed in Ref.8. For one thing, Bedford and Stern assume independent finite motions of each constituent while in Ref.8, as already stated, although the motion of the center of mass is taken to be finite, the relative motion of the separate constituents is taken to be infinitesimal. It is felt that the procedure employed in Ref.9 is physically unrealistic because the solid composite would rupture long before the relative displacements became large. Secondly, in Ref.9 a conservation of energy relation is written separately for each constituent while in Ref. 8 a single conservation of energy relation is written for the entire composite. When a separate conservation of energy relation is written for each constituent, the energy of interaction between the constituents is not included in the definition of the stored energy density nor do the associated rate terms appear in the formal expression for the first law of thermodynamics. As a consequence,

in Ref.9 different temperatures and entropy densities are defined and a separate rate of entropy production inequality is postulated for each constituent, whereas in Ref.8 one energy density, one temperature and one entropy density are defined and one rate of entropy production inequality is employed in the usual manner. As a result, in Ref. 9 with the exception of the defined volumetric interaction terms which are taken to depend on the constitutive variables associated with all the constituents, the other dependent constitutive variables for each constituent are taken to depend on the constitutive variables associated with that constituent only. On the other hand, in Ref.8 the resulting single thermodynamic equation for the combined continuum takes a form that indicates that all dependent constitutive variables, including the relative stresses associated with all the different combinations of constituents, should depend on the constitutive variables associated with all the combinations of constituents. Clearly, when the theory of Bedford and Stern is linearized[10], the aforementioned physical objection concerning the large relative motion of the constituents vanishes. However, it should be clear from the above discussion that certain intrinsic differences in the descriptions remain.

In this paper the above discussed model of interpenetrating solid continua is applied in obtaining a description of a three-constituent composite material. In the two-constituent case the model is identical to the one employed in Ref. 8 provided the electronic charge and spin continua are omitted and the ionic charge is ignored. In the three-constituent case the model is a straightforward generalization of the two-constituent case, and from there the generalization to N constituents is obvious. For obvious reasons, the general equations are determined only in the three-and N-constituent cases. The procedure employed in obtaining the description is exactly the same as in Ref.8 , but in the absence of the electromagnetic field. However, in this treatment

simple Kelvin-type viscous dissipation is included. As already indicated, the motion of the center of mass of any point of the entire composite may be finite while the relative displacement of any constituent from the mass center must be infinitesimal. Each constituent interacts with neighboring elements of the same constituent by means of a traction vector associated with that constituent. In addition, each constituent interacts with all other constituents at that point by means of volumetric interaction forces and couples, both of which are equal and opposite in pairs.

The application of the appropriate equations of balance of mass and momentum to the respective continua yields the material equations of motion, which, as usual, constitutes an underdetermined system. The application of the equation of the conservation of energy to the combined material continuum results in the first law of thermodynamics, which, with the aid of the second law of thermodynamics[11,12] and the principle of material objectivity[13,14], enables the determination of the constitutive equations of our nonlinear description of composite materials. These constitutive equations along with the aforementioned equations of motion and the thermodynamic dissipation equation result in a properly determined system, which can readily be reduced to $(3N+1)$ equations in $(3N+1)$ dependent variables. In order to complete the system of equations, jump (or boundary) conditions across moving, not necessarily material, surfaces of discontinuity are determined from the appropriate integral forms of the field equations. It should be mentioned at this point that the resulting system of nonlinear differential equations and boundary conditions is considered to be valid for fiber reinforced type composites, as well as some other types, but not for laminated composites unless the thickness of a lamina is small compared to a critical dimension or wavelength. It is felt that the equations provide a reasonable description of such materials as, say, fiber reinforced rubber.

If in a fiber reinforced composite the fiber is not continuous (chopped fiber), the description is simplified by neglecting the traction in the constituent of the model representing the discontinuous (chopped) fiber reinforcement and including only the volumetric interaction between this constituent and the matrix.

Since the resulting general nonlinear equations are relatively intractable for the treatment of many problems in their natural form, the linear version of the equations is extracted from the general one. These linear equations are specialized to the important case of a transversely isotropic material, which occurs very frequently in continuous fiber reinforced composites, and the fully isotropic case. The static potentials analogous to the Boussinesq Papkovitch potentials of classical elasticity are obtained from the linear equations for the two constituent isotropic composite. From this static potential representation the solutions for concentrated forces in the infinite two constituent isotropic composite are obtained. The dynamic potentials analogous to the Lamé potentials of classical elasticity are obtained from the linear equations for the two constituent isotropic composite. Completeness is established in both cases. Plane wave solutions of the linear equations for the two constituent composite are presented for the isotropic case. The solutions reveal the existence of two sets of waves, higher (in frequency) ones and lower ones, as expected. The lower ones are dispersive and approach the non-dispersive velocity of classical linear elasticity from below as the wavenumber approaches zero and the upper ones, which are highly dispersive, have non-zero cut-off frequencies corresponding to the aforementioned resonances at which the reinforcement might separate from the matrix.

Since the defined material constants are not known for any two constituent composite, it is suggested that plane wave measurements be made and correlated

with the above-mentioned plane wave solutions in order to obtain the defined linear elastic constants of the two constituent composite, in much the same manner as in anisotropic elastic (or piezoelectric) materials[15-17]. The theory can then readily be checked by comparing calculations with measurements in additional redundant directions. Moreover, when known material constants are available, such things as surface wave velocity dispersion can be calculated and compared with measurement. In addition, systematic dispersion information can be used for the non-destructive testing of fiber reinforced composite materials. An analysis of surface waves in a two constituent composite has been made and some calculations have been performed when the fiber reinforced composite material is simplified sufficiently that some of the constants can be estimated from the known elastic constants of the individual constituents of the composite. This investigation will be reported in a forthcoming work.

In closing the Introduction we note that more general single continuum theories, commonly referred to as microstructure theories[18-21], can be and, indeed, have been[22] applied to certain composites to describe some of the phenomena that the model of interpenetrating solid continua describes. However, the resulting equations are quite different from those presented here and we find it difficult to identify the physical meaning of the microstructure variables with any degree of certainty.

2. The Interacting Continua

As indicated in the Introduction, the macroscopic model we first consider consists essentially of three distinct interpenetrating solid continua. Initially, all continua occupy the same region of space and, hence, have the same material coordinates X_L. The motion of the center of mass of the combined continuum is described by the mapping[23]

$$y_i = y_i(X_L,t)\ ,\quad \underset{\sim}{y} = \underset{\sim}{y}(\underset{\sim}{X},t)\ , \tag{2.1}$$

which is one-to-one and differentiable as often as required. In (2.1) the y_i denote the spatial (or present) coordinates and X_L, the material (or reference) coordinates of the center of mass and t denotes the time. We consistently use the convention that capital indices denote the Cartesian components of $\underset{\sim}{X}$ and lower case indices, the Cartesian components of $\underset{\sim}{y}$. Thus, $\underset{\sim}{X}$ and $\underset{\sim}{y}$ denote the initial position of all material points and the center of mass of the combined continuum, respectively. Both dyadic and Cartesian tensor notation are used interchangeably. A comma followed by an index denotes partial differentiation with respect to a coordinate, i.e.,

$$y_{i,L} = \partial y_i/\partial X_L\ ,\quad X_{K,j} = \partial X_K/\partial y_j\ , \tag{2.2}$$

and the summation convention for repeated tensor indices is employed. The superscripts 1,2,3 are used to denote the respective continua. Since each continuum possesses a positive reference mass density $\rho_o^{(n)}$ $(n = 1,2,3)$ and initially occupies the same region of space, we have

$$\rho_o = \rho_o^{(1)} + \rho_o^{(2)} + \rho_o^{(3)}\ , \tag{2.3}$$

where ρ_o is the total reference mass density of the combined continuum.

In a (finite) motion each continuum is permitted to displace with respect to the center of mass of the combined continuum by <u>infinitesimal</u> displacement fields $\underset{\sim}{w}^{(1)}, \underset{\sim}{w}^{(2)}, \underset{\sim}{w}^{(3)}$. A schematic diagram indicating the motion of the model appears in Fig.1. The <u>infinitesimal</u> displacement fields $\underset{\sim}{w}^{(n)}$ are regarded as functions of $\underset{\sim}{y}$ and t. Since the $\underset{\sim}{w}^{(n)}$ are infinitesimal and

$$\underset{\sim}{y}^{(n)} = \underset{\sim}{y} + \underset{\sim}{w}^{(n)}(\underset{\sim}{y},t)\ , \tag{2.4}$$

and the determinant of a matrix product is equal to the product of the determinants, we have

$$v^{(n)} = V(1 + \underset{\sim}{\nabla}_{y} \cdot \underset{\sim}{w}^{(n)}) \approx V\,, \tag{2.5}$$

where $v^{(n)}$ is the present volume of the nth constituent, V is the present volume of the center of mass and

$$V = JV_o\,, \tag{2.6}$$

where V_o is the reference volume of the material and as usual

$$J = \det y_{i,L}\,. \tag{2.7}$$

Inasmuch as mass is conserved separately for each constituent, from (2.5) and (2.6) we have

$$\rho^{(n)}J = \rho_o^{(n)}\,, \tag{2.8}$$

which enables us to write

$$\rho = \rho^{(1)} + \rho^{(2)} + \rho^{(3)}\,, \tag{2.9}$$

where

$$\rho J = \rho_o\,, \tag{2.10}$$

and ρ is the total present mass density of the combined continuum. Since $\underset{\sim}{y}$ has been defined as the center of mass of the combined continuum, we may write

$$\int_{V^{(1)}} (\underset{\sim}{y} + \underset{\sim}{w}^{(1)})\rho^{(1)}dV + \int_{V^{(2)}} (\underset{\sim}{y} + \underset{\sim}{w}^{(2)})\rho^{(2)}\,dV +$$

$$\int_{V^{(3)}} (\underset{\sim}{y} + \underset{\sim}{w}^{(3)})\rho^{(3)}dV = \int_{V} (\rho^{(1)} + \rho^{(2)} + \rho^{(3)})\underset{\sim}{y}\,dV\,, \tag{2.11}$$

and by virtue of (2.5), we have

$$\rho^{(1)}\underset{\sim}{w}^{(1)} + \rho^{(2)}\underset{\sim}{w}^{(2)} + \rho^{(3)}\underset{\sim}{w}^{(3)} = 0\,.$$

In addition, because mass is conserved separately for each continuum, we further obtain

$$\rho^{(1)} \frac{d\underset{\sim}{w}^{(1)}}{dt} + \rho^{(2)} \frac{d\underset{\sim}{w}^{(2)}}{dt} + \rho^{(3)} \frac{d\underset{\sim}{w}^{(3)}}{dt} = 0 , \tag{2.13}$$

where d/dt denotes the material time derivative.

The interpenetrating continua interact with each other by means of defined local equal and opposite force fields ${}^{L}\underset{\sim}{F}^{12} = -{}^{L}\underset{\sim}{F}^{21}$, ${}^{L}\underset{\sim}{F}^{13} = -{}^{L}\underset{\sim}{F}^{31}$, ${}^{L}\underset{\sim}{F}^{23} = -{}^{L}\underset{\sim}{F}^{32}$, which are located at the position $\underset{\sim}{y}$, where the first superscript denotes the continuum being acted on and the second, the continuum producing the action, and defined equal and opposite local material couples ${}^{L}\underset{\sim}{C}^{12} = -{}^{L}\underset{\sim}{C}^{21}$, ${}^{L}\underset{\sim}{C}^{13} = -{}^{L}\underset{\sim}{C}^{31}$, ${}^{L}\underset{\sim}{C}^{23} = -{}^{L}\underset{\sim}{C}^{32}$. Each continuum interacts with neighboring elements of the same continuum by means of a traction force per unit area $\underset{\sim}{t}^{(1)}$, $\underset{\sim}{t}^{(2)}$, $\underset{\sim}{t}^{(3)}$ acting across the surface of separation. Schematic diagrams illustrating the above-mentioned interaction in the model are shown in Figs.2-4.

3. The Equations of Balance

In view of the discussion in Section 2, the equations of the conservation of mass for the different continua may be written in the form

$$\frac{d}{dt}\int_V \rho^{(1)}\, dV = 0 , \quad \frac{d}{dt}\int_V \rho^{(2)}\, dV = 0 , \quad \frac{d}{dt}\int_V \rho^{(3)}\, dV = 0 , \tag{3.1}$$

where V is an arbitrary element of volume for which each of the continua has the same material coordinates. From (2.3) and (3.1), we obtain the equation of the conservation of mass for the combined continuum in the form

$$\frac{d}{dt}\int_V \rho\, dV = 0 . \tag{3.2}$$

The equations of the conservation of linear momentum for each of the three continua are, respectively

$$\int_S \underset{\sim}{t}^{(1)}\,dS + \int_V \rho^{(1)}\underset{\sim}{f}^{(1)}\,dV + \int_V ({}^L\underset{\sim}{F}^{12} + {}^L\underset{\sim}{F}^{13})\,dV = \frac{d}{dt}\int_V \rho^{(1)}\left(\underset{\sim}{v} + \frac{d\underset{\sim}{w}^{(1)}}{dt}\right)dV\,, \qquad (3.3)$$

$$\int_S \underset{\sim}{t}^{(2)}\,dS + \int_V \rho^{(2)}\underset{\sim}{f}^{(2)}\,dV + \int_V ({}^L\underset{\sim}{F}^{21} + {}^L\underset{\sim}{F}^{23})\,dV = \frac{d}{dt}\int_V \rho^{(2)}\left(\underset{\sim}{v} + \frac{d\underset{\sim}{w}^{(2)}}{dt}\right)dV\,, \qquad (3,4)$$

$$\int_S \underset{\sim}{t}^{(3)}\,dS + \int_V \rho^{(3)}\underset{\sim}{f}^{(3)}\,dV + \int_V ({}^L\underset{\sim}{F}^{31} + {}^L\underset{\sim}{F}^{32})\,dV = \frac{d}{dt}\int_V \rho^{(3)}\left(\underset{\sim}{v} + \frac{d\underset{\sim}{w}^{(3)}}{dt}\right)dV\,, \qquad (3.5)$$

where $\underset{\sim}{v} = d\underset{\sim}{y}/dt$. The equations of the conservation of angular momentum for each of the three continua are, respectively

$$\int_S (\underset{\sim}{y} + \underset{\sim}{w}^{(1)}) \times \underset{\sim}{t}^{(1)}\,dS + \int_V (\underset{\sim}{y} + \underset{\sim}{w}^{(1)}) \times \rho^{(1)}\underset{\sim}{f}^{(1)}\,dV + \int_V ({}^L\underset{\sim}{C}^{12} + {}^L\underset{\sim}{C}^{13})\,dV$$
$$+ \int_V \underset{\sim}{y} \times ({}^L\underset{\sim}{F}^{12} + {}^L\underset{\sim}{F}^{13})\,dV = \frac{d}{dt}\int_V (\underset{\sim}{y} + \underset{\sim}{w}^{(1)}) \times \rho^{(1)}\left(\underset{\sim}{v} + \frac{d\underset{\sim}{w}^{(1)}}{dt}\right)dV\,, \qquad (3.6)$$

$$\int_S (\underset{\sim}{y} + \underset{\sim}{w}^{(2)}) \times \underset{\sim}{t}^{(2)}\,dS + \int_V (\underset{\sim}{y} + \underset{\sim}{w}^{(2)}) \times \rho^{(2)}\underset{\sim}{f}^{(2)}\,dV + \int_V ({}^L\underset{\sim}{C}^{21} + {}^L\underset{\sim}{C}^{23})\,dV$$
$$+ \int_V \underset{\sim}{y} \times ({}^L\underset{\sim}{F}^{21} + {}^L\underset{\sim}{F}^{23})\,dV = \frac{d}{dt}\int_V (\underset{\sim}{y} + \underset{\sim}{w}^{(2)}) \times \rho^{(2)}\left(\underset{\sim}{v} + \frac{d\underset{\sim}{w}^{(2)}}{dt}\right)dV\,, \qquad (3.7)$$

$$\int_S (\underset{\sim}{y} + \underset{\sim}{w}^{(3)}) \times \underset{\sim}{t}^{(3)}\,dS + \int_V (\underset{\sim}{y} + \underset{\sim}{w}^{(3)}) \times \rho^{(3)}\underset{\sim}{f}^{(3)}\,dV + \int_V ({}^L\underset{\sim}{C}^{31} + {}^L\underset{\sim}{C}^{32})\,dV$$
$$+ \int_V \underset{\sim}{y} \times ({}^L\underset{\sim}{F}^{31} + {}^L\underset{\sim}{F}^{32})\,dV = \frac{d}{dt}\int_V (\underset{\sim}{y} + \underset{\sim}{w}^{(3)}) \times \rho^{(3)}\left(\underset{\sim}{v} + \frac{d\underset{\sim}{w}^{(3)}}{dt}\right)dV\,. \qquad (3.8)$$

Application of (3.3) - (3.5) to an elementary tetrahedron in the usual manner yields the definition of the respective stress tensors of each of the three continua, thus

$$\underset{\sim}{t}^{(1)} = \underset{\sim}{n} \cdot \underset{\sim}{\tau}^{(1)}\,, \quad \underset{\sim}{t}^{(2)} = \underset{\sim}{n} \cdot \underset{\sim}{\tau}^{(2)}\,, \quad \underset{\sim}{t}^{(3)} = \underset{\sim}{n} \cdot \underset{\sim}{\tau}^{(3)}\,. \qquad (3.9)$$

Substituting from (3.9) into (3.3) - (3.5), respectively, taking the material time derivatives, using (3.1), the divergence theorem and the arbitrariness of V, we obtain

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{\tau}^{(1)} + \rho^{(1)} \underset{\sim}{f}^{(1)} - \rho^{(1)} \frac{d\underset{\sim}{v}}{dt} - \rho^{(1)} \frac{d^2 \underset{\sim}{w}^{(1)}}{dt^2} + {}^{L}\underset{\sim}{F}^{12} + {}^{L}\underset{\sim}{F}^{13} = 0 , \tag{3.10}$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{\tau}^{(2)} + \rho^{(2)} \underset{\sim}{f}^{(2)} - \rho^{(2)} \frac{d\underset{\sim}{v}}{dt} - \rho^{(2)} \frac{d^2 \underset{\sim}{w}^{(2)}}{dt^2} + {}^{L}\underset{\sim}{F}^{21} + {}^{L}\underset{\sim}{F}^{23} = 0 , \tag{3.11}$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{\tau}^{(3)} + \rho^{(3)} \underset{\sim}{f}^{(3)} - \rho^{(3)} \frac{d\underset{\sim}{v}}{dt} - \rho^{(3)} \frac{d^2 \underset{\sim}{w}^{(3)}}{dt^2} + {}^{L}\underset{\sim}{F}^{31} + {}^{L}\underset{\sim}{F}^{32} = 0 , \tag{3.12}$$

which are the stress equations of motion of the three continua, and where $\underset{\sim}{\nabla} = \underset{\sim}{e}_i \, \partial/\partial y_i$ and $\underset{\sim}{e}_i$ is a unit base vector in the ith Cartesian direction.

Substituting from (3.9) into (3.6) - (3.8), respectively, taking the material time derivatives, using (3.1), the divergence theorem, (3.10) - (3.12) and the arbitrariness of V, we obtain

$$\begin{aligned} &\underset{\sim}{e}_{\ell} e_{\ell ij} \tau_{ij}^{(1)} + \underset{\sim}{e}_{\ell} e_{\ell kj} (w_k^{(1)} \tau_{ij}^{(1)})_{,i} + \underset{\sim}{w}^{(1)} \times \rho^{(1)} \underset{\sim}{f}^{(1)} + {}^{L}\underset{\sim}{C}^{12} + {}^{L}\underset{\sim}{C}^{13} \\ &\qquad - \underset{\sim}{w}^{(1)} \times \rho^{(1)} \frac{d\underset{\sim}{v}}{dt} - \underset{\sim}{w}^{(1)} \times \rho^{(1)} \frac{d^2 \underset{\sim}{w}^{(1)}}{dt^2} = 0 , \end{aligned} \tag{3.13}$$

$$\begin{aligned} &\underset{\sim}{e}_{\ell} e_{\ell ij} \tau_{ij}^{(2)} + \underset{\sim}{e}_{\ell} e_{\ell kj} (w_k^{(2)} \tau_{ij}^{(2)})_{,i} + \underset{\sim}{w}^{(2)} \times \rho^{(2)} \underset{\sim}{f}^{(2)} + {}^{L}\underset{\sim}{C}^{21} + {}^{L}\underset{\sim}{C}^{23} \\ &\qquad - \underset{\sim}{w}^{(2)} \times \rho^{(2)} \frac{d\underset{\sim}{v}}{dt} - \underset{\sim}{w}^{(2)} \times \rho^{(2)} \frac{d^2 \underset{\sim}{w}^{(2)}}{dt^2} = 0 , \end{aligned} \tag{3.14}$$

$$\begin{aligned} &\underset{\sim}{e}_{\ell} e_{\ell ij} \tau_{ij}^{(3)} + \underset{\sim}{e}_{\ell} e_{\ell kj} (w_k^{(3)} \tau_{ij}^{(3)})_{,i} + w^{(3)} \times \rho^{(3)} f^{(3)} + {}^{L}\underset{\sim}{C}^{31} + {}^{L}\underset{\sim}{C}^{32} \\ &\qquad - \underset{\sim}{w}^{(3)} \times \rho^{(3)} \frac{d\underset{\sim}{v}}{dt} - \underset{\sim}{w}^{(3)} \times \rho^{(3)} \frac{d^2 \underset{\sim}{w}^{(3)}}{dt^2} = 0 , \end{aligned} \tag{3.15}$$

which constitute the equations of the conservation of angular momentum of each of the three respective continua.

Adding (3.10) - (3.12), we obtain

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{\tau} + \rho \underset{\sim}{f} = \rho \frac{d\underset{\sim}{v}}{dt} , \tag{3.16}$$

which are the stress equations of motion of the combined continuum, and

$$\underset{\sim}{T} = \underset{\sim}{T}^{(1)} + \underset{\sim}{T}^{(2)} + \underset{\sim}{T}^{(3)} \tag{3.17}$$

$$\rho\underset{\sim}{f} = \rho^{(1)}\underset{\sim}{f}^{(1)} + \rho^{(2)}\underset{\sim}{f}^{(2)} + \rho^{(3)}\underset{\sim}{f}^{(3)} \tag{3.18}$$

where $\underset{\sim}{T}$ is the total mechanical stress tensor and $\underset{\sim}{f}$ is the total body force per unit mass. Now let us define the constants $r^{(1)}$ and $r^{(2)}$ by

$$r^{(1)} = \rho^{(1)}/\rho^{(3)}, \quad r^{(2)} = \rho^{(2)}/\rho^{(3)}, \tag{3.19}$$

and then the subtraction of $r^{(1)}$ times (3.12) from (3.10) yields

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{D}^{(1)} + \rho^{(1)}\underset{\sim}{f}^{(1)} - \rho^{(1)} \frac{d^2\underset{\sim}{\eta}^{(1)}}{dt^2} + \underset{\sim}{\mathfrak{F}}^{(1)} = 0, \tag{3.20}$$

where

$$D_{ij}^{(1)} = T_{ij}^{(1)} - r^{(1)}T_{ij}^{(3)}, \tag{3.21}$$

$$\underset{\sim}{\mathfrak{F}}^{(1)} = {}^{L}\underset{\sim}{F}^{12} + {}^{L}\underset{\sim}{F}^{13} - r^{(1)}({}^{L}\underset{\sim}{F}^{31} + {}^{L}\underset{\sim}{F}^{32}), \tag{3.22}$$

$$\tilde{\underset{\sim}{f}}^{(1)} = \underset{\sim}{f}^{(1)} - \underset{\sim}{f}^{(3)}, \tag{3.23}$$

and

$$\underset{\sim}{\eta}^{(1)} = \underset{\sim}{w}^{(1)} - \underset{\sim}{w}^{(3)} = (1 + r^{(1)})\underset{\sim}{w}^{(1)} + r^{(2)}\underset{\sim}{w}^{(2)} \tag{3.24}$$

and we have employed (2.12). Similarly, subtracting $r^{(2)}$ times (3.12) from (3.11), we obtain

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{D}^{(2)} + \rho^{(2)}\underset{\sim}{f}^{(2)} - \rho^{(2)} \frac{d^2\underset{\sim}{\eta}^{(2)}}{dt^2} + \underset{\sim}{\mathfrak{F}}^{(2)} = 0, \tag{3.25}$$

where

$$D_{ij}^{(2)} = T_{ij}^{(2)} - r^{(2)}T_{ij}^{(3)}, \quad \underset{\sim}{\mathfrak{F}}^{(2)} = {}^{L}\underset{\sim}{F}^{21} + {}^{L}\underset{\sim}{F}^{23} - r^{(2)}({}^{L}\underset{\sim}{F}^{31} + {}^{L}\underset{\sim}{F}^{32}),$$

$$\tilde{\underset{\sim}{f}}^{(2)} = \underset{\sim}{f}^{(2)} - \underset{\sim}{f}^{(3)}, \quad \underset{\sim}{\eta}^{(2)} = \underset{\sim}{w}^{(2)} - \underset{\sim}{w}^{(3)} = (1 + r^{(2)})\underset{\sim}{w}^{(2)} + r^{(1)}\underset{\sim}{w}^{(1)}. \tag{3.26}$$

Equations (3.20) and (3.25) are called the difference or relative equations of motion, and $D_{ij}^{(1)}$, $D_{ij}^{(2)}$ and $\underset{\sim}{\eta}^{(1)}$, $\underset{\sim}{\eta}^{(2)}$ are the difference stresses and difference

displacements, respectively. Adding (3.13) - (3.15) and employing (2.12), (3.17), (3.19) - (3.21) and (3.23) - (3.26), we obtain

$$\underset{\sim}{e}_{\ell} e_{\ell ij} \tau_{ij} + \underset{\sim}{e}_{\ell} e_{\ell kj} (w_k^{(1)} D_{ij}^{(1)} + w_k^{(2)} D_{ij}^{(2)})_{,i}$$

$$- \underset{\sim}{w}^{(1)} \times (\underset{\sim}{\nabla} \cdot \underset{\sim}{D}^{(1)} + \underset{\sim}{\mathfrak{F}}^{(1)}) - \underset{\sim}{w}^{(2)} \times (\underset{\sim}{\nabla} \cdot \underset{\sim}{D}^{(2)} + \underset{\sim}{\mathfrak{F}}^{(2)}) = 0 , \qquad (3.27)$$

which is the equation of the conservation of angular momentum for the combined continuum. Equation (3.27) turns out to be of considerable value and interest when viscous type dissipation is considered. However, in the absence of viscous dissipation Eq,(3.27) is a direct consequence of the invariance of the stored energy function in a rigid rotation.

Although we cannot explicitly evaluate each of the defined couples of interaction ${}^{L}\underset{\sim}{C}^{mn}$ between the respective continua in the description presented here, we can readily evaluate the total internal couple acting on each continuum, i.e., the ${}^{L}\underset{\sim}{C}^{(n)}$, where

$${}^{L}\underset{\sim}{C}^{(1)} = {}^{L}\underset{\sim}{C}^{12} + {}^{L}\underset{\sim}{C}^{13} , \quad {}^{L}\underset{\sim}{C}^{(2)} = {}^{L}\underset{\sim}{C}^{21} + {}^{L}\underset{\sim}{C}^{23} , \quad {}^{L}\underset{\sim}{C}^{31} = {}^{L}\underset{\sim}{C}^{31} + {}^{L}\underset{\sim}{C}^{32} , \qquad (3.28)$$

and that is all that is required in this type of description. Specifically, the ${}^{L}\underset{\sim}{C}^{(n)}$ may be determined <u>a posteriori</u> from (3.17), (3.21), $(3.26)_1$ and (3.13) - (3.15). Similar statements hold in the case of the defined forces of interaction ${}^{L}\underset{\sim}{F}^{mn}$ between the respective continua, and we can readily evaluate <u>a posteriori</u> the total internal force ${}^{L}\underset{\sim}{\Gamma}^{(n)}$ acting on each continuum from (3.17), (3.21), $(3.26)_1$ and (3.10) - (3.12), where

$${}^{L}\underset{\sim}{\Gamma}^{(1)} = {}^{L}\underset{\sim}{F}^{12} + {}^{L}\underset{\sim}{F}^{13} , \quad {}^{L}\underset{\sim}{\Gamma}^{(2)} = {}^{L}\underset{\sim}{F}^{21} + {}^{L}\underset{\sim}{F}^{23} , \quad {}^{L}\underset{\sim}{\Gamma}^{(3)} = {}^{L}\underset{\sim}{F}^{31} + {}^{L}\underset{\sim}{F}^{32} . \qquad (3.29)$$

4. Thermodynamic Considerations

The conservation of energy for the combined material continuum can be written in the form

$$\frac{d}{dt}\int_V (T + \rho\epsilon)\, dV = \int_S \left[\underset{\sim}{t}^{(1)} \cdot \left(\underset{\sim}{v} + \frac{d\underset{\sim}{w}^{(1)}}{dt}\right) + \underset{\sim}{t}^{(2)} \cdot \left(\underset{\sim}{v} + \frac{d\underset{\sim}{w}^{(2)}}{dt}\right) + \underset{\sim}{t}^{(3)} \cdot \left(\underset{\sim}{v} + \frac{d\underset{\sim}{w}^{(3)}}{dt}\right) - \underset{\sim}{n}\cdot\underset{\sim}{q}\right] dS + \int_V \left[\rho^{(1)}\underset{\sim}{f}^{(1)} \cdot \left(\underset{\sim}{v} + \frac{d\underset{\sim}{w}^{(1)}}{dt}\right) + \rho^{(2)}\underset{\sim}{f}^{(2)} \cdot \left(\underset{\sim}{v} + \frac{d\underset{\sim}{w}^{(2)}}{dt}\right) + \rho^{(3)}\underset{\sim}{f}^{(3)} \cdot \left(\underset{\sim}{v} + \frac{d\underset{\sim}{w}^{(3)}}{dt}\right)\right] dV, \tag{4.1}$$

where T is the kinetic energy per unit volume, ϵ is the internal stored energy per unit mass, $\underset{\sim}{t}^{(n)} \cdot \left(\underset{\sim}{v} + \frac{d\underset{\sim}{w}^{(n)}}{dt}\right)$, $n = 1,2,3$, denote the rates of working per unit area of the mechanical surface tractions acting in the three continua, respectively, $\underset{\sim}{n}\cdot\underset{\sim}{q}$ is the rate of efflux of heat per unit area and $\rho^{(n)}\underset{\sim}{f}^{(n)} \cdot \left(\underset{\sim}{v} + \frac{d\underset{\sim}{w}^{(n)}}{dt}\right)$ denote the rates of working per unit volume of body forces acting in three continua, respectively. In order to obtain expressions for T, we must return to our model of the combined material continuum.

From the model of the continuum it is clear that the kinetic energy per unit volume is of the form

$$T = \frac{1}{2}\left[\rho^{(1)}\left(\underset{\sim}{v} + \frac{d\underset{\sim}{w}^{(1)}}{dt}\right) \cdot \left(\underset{\sim}{v} + \frac{d\underset{\sim}{w}^{(1)}}{dt}\right) + \rho^{(2)}\left(\underset{\sim}{v} + \frac{d\underset{\sim}{w}^{(2)}}{dt}\right) \cdot \left(\underset{\sim}{v} + \frac{d\underset{\sim}{w}^{(2)}}{dt}\right) + \rho^{(3)}\left(\underset{\sim}{v} + \frac{d\underset{\sim}{w}^{(3)}}{dt}\right) \cdot \left(\underset{\sim}{v} + \frac{d\underset{\sim}{w}^{(3)}}{dt}\right)\right]. \tag{4.2}$$

Expanding terms in (4.2) and employing (2.9), (2.13) and (3.19), we obtain

$$T = \frac{1}{2}\left[\rho\, \underset{\sim}{v}\cdot\underset{\sim}{v} + \rho^{(1)}(1 + r^{(1)})\, \frac{d\underset{\sim}{w}^{(1)}}{dt} \cdot \frac{d\underset{\sim}{w}^{(1)}}{dt} + \rho^{(2)}(1 + r^{(2)})\, \frac{d\underset{\sim}{w}^{(2)}}{dt} \cdot \frac{d\underset{\sim}{w}^{(2)}}{dt} + 2r^{(1)}\rho^{(2)}\, \frac{d\underset{\sim}{w}^{(1)}}{dt} \cdot \frac{d\underset{\sim}{w}^{(2)}}{dt}\right]. \tag{4.3}$$

Substituting from (2.13), (3.18), (3.23), (3.26) and (4.3) into (4.1), we obtain

$$\frac{d}{dt}\int_V \left[\frac{1}{2}\rho\, \underset{\sim}{v}\cdot\underset{\sim}{v} + \frac{1}{2}\rho^{(1)}(1+r^{(1)})\frac{d\underset{\sim}{w}^{(1)}}{dt}\cdot\frac{d\underset{\sim}{w}^{(1)}}{dt}\right.$$
$$\left. + \frac{1}{2}\rho^{(2)}\frac{d\underset{\sim}{w}^{(2)}}{dt}\cdot\frac{d\underset{\sim}{w}^{(2)}}{dt}(1+r^{(2)}) + r^{(1)}\rho^{(2)}\frac{d\underset{\sim}{w}^{(1)}}{dt}\cdot\frac{d\underset{\sim}{w}^{(2)}}{dt} + \rho\epsilon\right] dV =$$
$$\int_S \left(\underset{\sim}{t}\cdot\underset{\sim}{v} + \underset{\sim}{d}^{(1)}\cdot\frac{d\underset{\sim}{w}^{(1)}}{dt} + \underset{\sim}{d}^{(2)}\cdot\frac{d\underset{\sim}{w}^{(2)}}{dt} - \underset{\sim}{n}\cdot\underset{\sim}{q}\right) dS$$
$$+ \int_V \left(\rho\underset{\sim}{f}\cdot\underset{\sim}{v} + \rho^{(1)}\tilde{\underset{\sim}{f}}^{(1)}\cdot\frac{d\underset{\sim}{w}^{(1)}}{dt} + \rho^{(2)}\tilde{\underset{\sim}{f}}^{(2)}\cdot\frac{d\underset{\sim}{w}^{(2)}}{dt}\right) dV\,, \tag{4.4}$$

where

$$\underset{\sim}{t} = \underset{\sim}{t}^{(1)} + \underset{\sim}{t}^{(2)} + \underset{\sim}{t}^{(3)}\,,\quad \underset{\sim}{d}^{(1)} = \underset{\sim}{t}^{(1)} - r^{(1)}\underset{\sim}{t}^{(3)}\,,\quad \underset{\sim}{d}^{(2)} = \underset{\sim}{t}^{(2)} - r^{(2)}\underset{\sim}{t}^{(3)}\,, \tag{4.5}$$

and from (3.9), (3.17), (3.21) and $(3.26)_1$, we have

$$\underset{\sim}{t} = \underset{\sim}{n}\cdot\underset{\sim}{\tau}\,,\quad \underset{\sim}{d}^{(1)} = \underset{\sim}{n}\cdot\underset{\sim}{D}^{(1)}\,,\quad \underset{\sim}{d}^{(2)} = \underset{\sim}{n}\cdot\underset{\sim}{D}^{(2)}\,, \tag{4.6}$$

and $\underset{\sim}{t}$ is the mechanical traction vector of the combined continuum, and the $\underset{\sim}{d}^{(n)}$, $n = 1, 2$, may be thought of as the difference traction vectors. Taking the material time derivative in (4.4) and using (4.6), (3.16), (3.20), (3.25) and employing the divergence theorem, the material time derivatives of (2.8) and (2.13) and the arbitrariness of V, we obtain

$$\rho\frac{d\epsilon}{dt} = \tau_{ij}v_{j,i} + D^{(1)}_{ij}\left(\frac{dw^{(1)}_j}{dt}\right)_{,i} + D^{(2)}_{ij}\left(\frac{dw^{(2)}_j}{dt}\right)_{,i}$$
$$- \underset{\sim}{\mathfrak{F}}^{(1)}\cdot\frac{d\underset{\sim}{w}^{(1)}}{dt} - \underset{\sim}{\mathfrak{F}}^{(2)}\cdot\frac{d\underset{\sim}{w}^{(2)}}{dt} - q_{i,i}\,, \tag{4.7}$$

which is the first law of thermodynamics for our combined continuum.

We may now introduce dissipation by assuming that the symmetric part of the total stress tensor $\underset{\sim}{\tau}^S$, the two difference stress tensors $\underset{\sim}{D}^{(1)}$ and $\underset{\sim}{D}^{(2)}$

and the two difference internal forces $\underset{\sim}{\mathfrak{J}}^{(1)}$ and $\underset{\sim}{\mathfrak{J}}^{(2)}$ may be written as a sum of a dissipative and a nondissipative part. This is a restrictive assumption, but it is believed to be adequate for our purposes[24]. Thus, we write

$$\underset{\sim}{\tau}^S = {}^R\underset{\sim}{\tau}^S + {}^D\underset{\sim}{\tau}^S , \quad \underset{\sim}{D}^{(1)} = {}^R\underset{\sim}{D}^{(1)} + {}^D\underset{\sim}{D}^{(1)} , \quad \underset{\sim}{D}^{(2)} = {}^R\underset{\sim}{D}^{(2)} + {}^D\underset{\sim}{D}^{(2)} ,$$

$$\underset{\sim}{\mathfrak{J}}^{(1)} = {}^R\underset{\sim}{\mathfrak{J}}^{(1)} + {}^D\underset{\sim}{\mathfrak{J}}^{(1)} , \quad \underset{\sim}{\mathfrak{J}}^{(2)} = {}^R\underset{\sim}{\mathfrak{J}}^{(2)} + {}^D\underset{\sim}{\mathfrak{J}}^{(2)} , \tag{4.8}$$

and in each case the superscript R indicates the nondissipative (stored energy) portion and the superscript D, the dissipative portion. Substituting from (4.8) into (3.27) and obtaining the tensor form from the vector form, we obtain

$$\tau^A_{ij} = {}^R\tau^A_{ij} + {}^D\tau^A_{ij} , \tag{4.9}$$

where

$$\begin{aligned} {}^R\tau^A_{ij} = \frac{1}{2}\Big[& {}^R D^{(1)}_{ki} w^{(1)}_{j,k} - {}^R D^{(1)}_{kj} w^{(1)}_{i,k} - {}^R\mathfrak{J}^{(1)}_i w^{(1)}_j + {}^R\mathfrak{J}^{(1)}_j w^{(1)}_i \\ & + {}^R D^{(2)}_{ki} w^{(2)}_{j,k} - {}^R D^{(2)}_{kj} w^{(2)}_{i,k} - {}^R\mathfrak{J}^{(2)}_i w^{(2)}_j + {}^R\mathfrak{J}^{(2)}_j w^{(2)}_i \Big] , \\ {}^D\tau^A_{ij} = \frac{1}{2}\Big[& {}^D D^{(1)}_{ki} w^{(1)}_{j,k} - {}^D D^{(1)}_{kj} w^{(1)}_{i,k} - {}^D\mathfrak{J}^{(1)}_i w^{(1)}_j + {}^D\mathfrak{J}^{(1)}_j w^{(1)}_i \\ & + {}^D D^{(2)}_{ki} w^{(2)}_{j,k} - {}^D D^{(2)}_{kj} w^{(2)}_{i,k} - {}^D\mathfrak{J}^{(2)}_i w^{(2)}_j + {}^D\mathfrak{J}^{(2)}_j w^{(2)}_i \Big] . \end{aligned} \tag{4.10}$$

Since $\underset{\sim}{\tau} = \underset{\sim}{\tau}^S + \underset{\sim}{\tau}^A$, from $(4.8)_1$ and (4.9), we may write

$$\underset{\sim}{\tau} = {}^R\underset{\sim}{\tau} + {}^D\underset{\sim}{\tau}^S + {}^D\underset{\sim}{\tau}^A , \tag{4.11}$$

where

$${}^R\underset{\sim}{\tau} = {}^R\underset{\sim}{\tau}^S + {}^R\underset{\sim}{\tau}^A . \tag{4.12}$$

Equation (4.11) is the form we are interested in employing because it enables us to obtain all the results of interest to us most readily. Substituting from $(4.8)_{2-5}$ and (4.11) into (4.7) and employing (4.10), we obtain

$$\rho \frac{d\varepsilon}{dt} = {}^{R}T_{ij} v_{j,i} + {}^{R}D^{(1)}_{ij} \left(\frac{dw^{(1)}_j}{dt}\right)_{,i} - {}^{R}\mathfrak{F}^{(1)}_j \frac{dw^{(1)}_j}{dt} + {}^{R}D^{(2)}_{ij} \left(\frac{dw^{(2)}_j}{dt}\right)_{,i}$$

$$- {}^{R}\mathfrak{F}^{(2)}_j \frac{dw^{(2)}_j}{dt} + {}^{D}T^{S}_{ij} d_{ij} + {}^{D}D^{(1)}_{kj} \left[\left(\frac{dw^{(1)}_j}{dt}\right)_{,k} - w^{(1)}_{i,k}\omega_{ij}\right]$$

$$- {}^{D}\mathfrak{F}^{(1)}_j \left[\frac{dw^{(1)}_j}{dt} - w^{(1)}_i \omega_{ij}\right] + {}^{D}D^{(2)}_{kj} \left[\left(\frac{dw^{(2)}_j}{dt}\right)_{,k} - w^{(2)}_{i,k}\omega_{ij}\right]$$

$$- {}^{D}\mathfrak{F}^{(2)}_j \left[\frac{dw^{(2)}_j}{dt} - w^{(2)}_i \omega_{ij}\right] - q_{i,i} \ , \qquad (4.13)$$

where d_{ij} and ω_{ij} are the rate of deformation and spin tensors, respectively, which are defined by[25]

$$d_{ij} = \frac{1}{2} (v_{j,i} + v_{i,j}) \ , \quad \omega_{ij} = \frac{1}{2} (v_{j,i} - v_{i,j}) \ . \qquad (4.14)$$

For the circumstances we have outlined, the mathematical expression of the second law of thermodynamics may be written in the form[26-28]

$$\rho \frac{d\varepsilon}{dt} - {}^{R}T_{ij} v_{j,i} - {}^{R}D^{(1)}_{ij} \left(\frac{dw^{(1)}_j}{dt}\right)_{,i} + {}^{R}\mathfrak{F}^{(1)}_j \frac{dw^{(1)}_j}{dt} - {}^{R}D^{(2)}_{ij} \left(\frac{dw^{(2)}_j}{dt}\right)_{,i}$$

$$+ {}^{R}\mathfrak{F}^{(2)}_j \frac{dw^{(2)}_j}{dt} = \rho\theta \frac{d\eta}{dt} \ , \qquad (4.15)$$

where θ is the positive absolute temperature and η is the entropy per unit mass. From (4.13) and (4.15), we have the dissipation equation

$$^{D}T^{S}_{ij} d_{ij} + {}^{D}D^{(1)}_{kj} \left[\left(\frac{dw^{(1)}_j}{dt}\right)_{,k} - w^{(1)}_{i,k}\omega_{ij}\right] - {}^{D}\mathfrak{F}^{(1)}_j \left[\frac{dw^{(1)}_j}{dt} - w^{(1)}_i \omega_{ij}\right]$$

$$+ {}^{D}D^{(2)}_{kj} \left[\left(\frac{dw^{(2)}_j}{dt}\right)_{,k} - w^{(2)}_{i,k}\omega_{ij}\right] - {}^{D}\mathfrak{F}^{(2)}_j \left[\frac{dw^{(2)}_j}{dt} - w^{(2)}_i \omega_{ij}\right] - q_{i,i} = \rho\theta \frac{d\eta}{dt} \ , \qquad (4.16)$$

and the entropy inequality may be written in the form

$$\rho \frac{d\eta}{dt} + \left(\frac{q_i}{\theta}\right)_{,i} = \frac{1}{\theta}\Big[{}^{D}T^{S}_{ij}d_{ij} + {}^{D}D^{(1)}_{kj}\Big(\Big(\frac{d}{dt}w^{(1)}_j\Big)_{,k} - w^{(1)}_{i,k}\omega_{ij}\Big)$$

$$- {}^{D}\mathfrak{Z}^{(1)}_j\Big(\frac{dw^{(1)}_j}{dt} - w^{(1)}_i\omega_{ij}\Big) + {}^{D}D^{(2)}_{kj}\Big(\Big(\frac{dw^{(2)}_j}{dt}\Big)_{,k} - w^{(2)}_{i,k}\omega_{ij}\Big)$$

$$- {}^{D}\mathfrak{Z}^{(2)}_j\Big(\frac{dw^{(2)}_j}{dt} - w^{(2)}_i\omega_{ij}\Big) - \frac{1}{\theta}q_i\theta_{,i}\Big] = \rho\Gamma \geq 0 \ , \tag{4.17}$$

where Γ is the positive rate of entropy production. At this point it should be noted that this theory can readily be generalized[29,30] to account for more general functional constitutive response in the manner set forth in a previous paper[7].

5. Constitutive Equations

Since we are concerned with thermodynamic processes for which both the state function equation (4.15) and the dissipation equation (4.16) are valid, we may determine the dissipative constitutive equations from (4.17) and the non-dissipative constitutive equations from (4.15) which, by virtue of the relations

$$v_{j,i} = X_{M,i}\frac{d}{dt}(y_{j,M}) \ , \quad \Big(\frac{dw^{(1)}_j}{dt}\Big)_{,i} = X_{M,i}\frac{d}{dt}(w^{(1)}_{j,M}) \ , \quad \Big(\frac{dw^{(2)}_j}{dt}\Big)_{,i} = X_{M,i}\frac{d}{dt}(w^{(2)}_{j,M}) \ ,$$

may be written in the form

$$\rho\frac{d\varepsilon}{dt} = {}^{R}T_{ij}X_{M,i}\frac{d}{dt}(y_{j,M}) + {}^{R}D^{(1)}_{ij}X_{M,i}\frac{d}{dt}(w^{(1)}_{j,M}) + {}^{R}D^{(2)}_{ij}X_{M,i}\frac{d}{dt}(w^{(2)}_{j,M})$$

$$- {}^{R}\mathfrak{Z}^{(1)}_j\frac{dw^{(1)}_j}{dt} - {}^{R}\mathfrak{Z}^{(2)}_j\frac{dw^{(2)}_j}{dt} + \rho\theta\frac{d\eta}{dt} \ . \tag{5.1}$$

Since the entropy inequality is of the form shown in (4.17), it turns out to be convenient to define the thermodynamic function ψ by the Legendre transformation

$$\psi = \varepsilon - \eta\theta \ . \tag{5.2}$$

The substitution of the material time derivative of (5.2) into (5.1) yields

$$\rho \frac{d\psi}{dt} = {}^{R}\tau_{ij} X_{M,i} \frac{d}{dt}(y_{j,M}) + {}^{R}D_{ij}^{(1)} X_{M,i} \frac{d}{dt}(w_{j,M}^{(1)}) + {}^{R}D_{ij}^{(2)} X_{M,i} \frac{d}{dt}(w_{j,M}^{(2)})$$
$$- {}^{R}\mathcal{F}_j^{(1)} \frac{dw_j^{(1)}}{dt} - {}^{R}\mathcal{F}_j^{(2)} \frac{dw_j^{(2)}}{dt} - \rho\eta \frac{d\theta}{dt} . \qquad (5.3)$$

Since (5.3) is a state function equation, we must have

$$\psi = \psi(y_{j,M}; w_{j,M}^{(1)}; w_{j,M}^{(2)}; w_j^{(1)}; w_j^{(2)}; \theta) . \qquad (5.4)$$

Substituting the material time derivative of (5.4) into (5.3), we obtain

$$\left({}^{R}\tau_{ij} X_{M,i} - \rho \frac{\partial\psi}{\partial(y_{j,M})}\right) \frac{d}{dt}(y_{j,M}) + \left({}^{R}D_{ij}^{(1)} X_{M,i} - \rho \frac{\partial\psi}{\partial(w_{j,M}^{(1)})}\right) \frac{d}{dt}(w_{j,M}^{(1)})$$
$$+ \left({}^{R}D_{ij}^{(2)} X_{M,i} - \rho \frac{\partial\psi}{\partial(w_{j,M}^{(2)})}\right) \frac{d}{dt}(w_{j,M}^{(2)}) - \left({}^{R}\mathcal{F}_j^{(1)} + \rho \frac{\partial\psi}{\partial w_j^{(1)}}\right) \frac{d}{dt} w_j^{(1)}$$
$$- \left({}^{R}\mathcal{F}_j^{(2)} + \rho \frac{\partial\psi}{\partial w_j^{(2)}}\right) \frac{d}{dt} w_j^{(2)} - \rho\left(\eta + \frac{\partial\psi}{\partial\theta}\right) \frac{d\theta}{dt} = 0 . \qquad (5.5)$$

Since (5.5) holds for arbitrary $d(y_{i,M})/dt$, $d(w_{j,M}^{(1)})/dt$, $d(w_{j,M}^{(2)})/dt$, $dw_j^{(1)}/dt$, $dw_j^{(2)}/dt$ and $d\theta/dt$, we have

$$X_{M,i}\, {}^{R}\tau_{ij} = \rho\, \partial\psi/\partial(y_{j,M}) , \quad {}^{R}\mathcal{F}_j^{(1)} = -\rho\, \partial\psi/\partial w_j^{(1)} , \qquad (5.6)$$

$$X_{M,i}\, {}^{R}D_{ij}^{(1)} = \rho\, \partial\psi/\partial(w_{j,M}^{(1)}) , \quad {}^{R}\mathcal{F}_j^{(2)} = -\rho\, \partial\psi/\partial w_j^{(2)} , \qquad (5.7)$$

$$X_{M,i}\, {}^{R}D_{ij}^{(2)} = \rho\, \partial\psi/\partial(w_{j,M}^{(2)}) , \quad \eta = -\rho\, \partial\psi/\partial\theta . \qquad (5.8)$$

Solving $(5.6)_1$, $(5.7)_1$ and $(5.8)_1$ for ${}^{R}\tilde{\tau}$, ${}^{R}\tilde{D}^{(1)}$ and ${}^{R}\tilde{D}^{(2)}$, respectively, we find

$${}^{R}\tau_{ij} = \rho y_{i,M} \frac{\partial\psi}{\partial(y_{j,M})} , \quad {}^{R}D_{ij}^{(1)} = \rho y_{i,M} \frac{\partial\psi}{\partial(w_{j,M}^{(1)})} , \quad {}^{R}D_{ij}^{(2)} = \rho y_{i,M} \frac{\partial\psi}{\partial(w_{j,M}^{(2)})} . \qquad (5.9)$$

Clearly, ψ cannot be an <u>arbitrary</u> function of the variables shown in (5.4) because in order to satisfy the principle of material objectivity[13,14] ϵ and,

hence, ψ must be scalar invariant under rigid rotations of the deformed body, and any arbitrary function of the 34 assumed variables (11 vectors and a scalar at the point y_k) will not be so invariant. However, there is a theorem on rotationally invariant functions of several vectors due to Cauchy[31], which says that ψ may be an arbitrary single-valued function of the scalar products of the vectors and the determinants of their components taken three at a time. Application of this theorem shows that ψ is expressible as an arbitrary function of 66 scalar products and 127 determinants and θ for a total of 194 quantities. However, the 194 quantities are not functionally independent and it is relatively easy to show, by using procedures similar to those employed in Section 6 of Ref. 6 that the 194 variables are expressible in terms of the 31 arguments

$$C_{KL} = y_{i,K}y_{i,L}\,,\quad P^{(1)}_{LM} = y_{k,L}w^{(1)}_{k,M}\quad N^{(1)}_{L} = y_{k,L}w^{(1)}_{k}\,,\quad \theta\,,\quad P^{(2)}_{LM} = y_{k,L}w^{(2)}_{k,M}\,,$$
$$N^{(2)}_{L} = y_{k,L}w^{(2)}_{k}\quad . \tag{5.10}$$

Thus, we find that ψ is invariant in a rigid rotation if it is a single-valued function of the 31 arguments listed in (5.10). Hence ψ may be reduced to the form

$$\psi = \psi(E_{KL}, N^{(1)}_{L}, N^{(2)}_{L}, P^{(1)}_{LM}, P^{(2)}_{LM}, \theta)\,, \tag{5.11}$$

in place of the form shown in (5.4), and where we have taken the liberty of replacing Green's deformation tensor C_{KL}, which does not vanish in the undeformed state, by the equivalent material strain tensor E_{KL}, which does vanish in the undeformed state, and is related to C_{KL} by[32]

$$E_{KL} = \frac{1}{2}\,(C_{KL} - \delta_{KL})\,. \tag{5.12}$$

From $(5.6)_2$, $(5.7)_2$, $(5.8)_2$ and (5.9) - (5.12), we obtain

$$
{}^{R}\tau_{ij} = \rho y_{i,L} y_{j,M} \frac{\partial\psi}{\partial E_{LM}} + \rho y_{i,L} \frac{\partial\psi}{\partial N_L^{(1)}} w_j^{(1)} + \rho y_{i,L} \frac{\partial\psi}{\partial N_L^{(2)}} w_j^{(2)}
$$

$$
+ \rho y_{i,L} \frac{\partial\psi}{\partial P_{LM}^{(1)}} w_{j,M}^{(1)} + \rho y_{i,L} \frac{\partial\psi}{\partial P_{LM}^{(2)}} w_{j,M}^{(2)} , \tag{5.13}
$$

$$
{}^{R}D_{ij}^{(1)} = \rho y_{i,M} y_{j,L} \frac{\partial\psi}{\partial P_{LM}^{(1)}} , \quad {}^{R}D_{ij}^{(2)} = \rho y_{i,M} y_{j,L} \frac{\partial\psi}{\partial P_{LM}^{(2)}} \tag{5.14}
$$

$$
{}^{R}\mathcal{J}_j^{(1)} = - \rho y_{j,L} \frac{\partial\psi}{\partial N_L^{(1)}} , \quad {}^{R}\mathcal{J}_j^{(2)} = - \rho y_{j,L} \frac{\partial\psi}{\partial N_L^{(2)}} , \quad \eta = - \rho \frac{\partial\psi}{\partial\theta} , \tag{5.15}
$$

where we have introduced the conventions $\partial\psi/\partial E_{LM} = \partial\psi/\partial E_{ML}$ and it is to be assumed that $\partial E_{KL}/\partial E_{LK} = 0$ in differentiating ψ. Substituting from (5.14) and (5.15) into (5.13) and employing the chain rule of differentiation, we obtain

$$
{}^{R}\tau_{ij} = \rho y_{i,L} y_{j,M} \frac{\partial\psi}{\partial E_{LM}} - {}^{R}\mathcal{J}_i^{(1)} w_j^{(1)} - {}^{R}\mathcal{J}_i^{(2)} w_j^{(2)}
$$

$$
+ {}^{R}D_{ki}^{(1)} w_{j,k}^{(1)} + {}^{R}D_{ki}^{(2)} w_{j,k}^{(2)} . \tag{5.16}
$$

Clearly, the antisymmetric part of ${}^{R}\tau_{ij}$ obtained from (5.16) is identical with the expression for ${}^{R}\tau_{ij}^{A}$ given in (4.10). Thus, even in this rather complex situation, the antisymmetric portion of the nondissipative part of the stress tensor is derivable from a thermodynamic state function and has just the value required by the conservation of angular momentum.

This brings us to a consideration of the dissipative constitutive equations, which are obtained from the entropy inequality (4.17) which may be written in the form

$$
{}^{D}\tau_{ij}^{S} d_{ij} + {}^{D}D_{kj}^{(1)} \zeta_{kj}^{(1)} - {}^{D}\mathcal{J}_j^{(1)} \beta_j^{(1)} + {}^{D}D_{kj}^{(2)} \zeta_{kj}^{(2)} - {}^{D}\mathcal{J}_j^{(2)} \beta_j^{(2)} - q_i \theta_{,i}/\theta^2 \geq 0 , \tag{5.17}
$$

where

$$\zeta^{(1)}_{kj} = (dw^{(1)}_j/dt)_{,k} - w^{(1)}_{i,k}\omega_{ij}\,, \quad \beta^{(1)}_j = dw^{(1)}_j/dt - w^{(1)}_i\omega_{ij}\,,$$

$$\zeta^{(2)}_{kj} = (dw^{(2)}_j/dt)_{,k} - w^{(2)}_{i,k}\omega_{ij}\,, \quad \beta^{(2)}_j = dw^{(2)}_j/dt - w^{(2)}_i\omega_{ij}\,. \tag{5.18}$$

Motivated by (5.17) we take the dissipative constitutive equations in the form

$$\begin{aligned}
{}^D\tau^S_{ij} &= {}^D\tau^S_{ij}(d_{k\ell},\ \zeta^{(1)}_{k\ell},\ \beta^{(1)}_k,\ \zeta^{(2)}_{k\ell},\ \beta^{(2)}_k,\ \theta_{,k})\,,\\
{}^D D^{(1)}_{ij} &= {}^D D^{(1)}_{ij}(d_{k\ell},\ \zeta^{(1)}_{k\ell},\ \beta^{(1)}_k,\ \zeta^{(2)}_{k\ell},\ \beta^{(2)}_k,\ \theta_{,k})\,,\\
{}^D \mathcal{J}^{(1)}_{j} &= {}^D \mathcal{J}^{(1)}_{j}(d_{k\ell},\ \zeta^{(1)}_{k\ell},\ \beta^{(1)}_k,\ \zeta^{(2)}_{k\ell},\ \beta^{(2)}_k,\ \theta_{,k})\,,\\
{}^D D^{(2)}_{ij} &= {}^D D^{(2)}_{ij}(d_{k\ell},\ \zeta^{(1)}_{k\ell},\ \beta^{(1)}_k,\ \zeta^{(2)}_{k\ell},\ \beta^{(2)}_k,\ \theta_{,k})\,,\\
{}^D \mathcal{J}^{(2)}_{j} &= {}^D \mathcal{J}^{(2)}_{j}(d_{k\ell},\ \zeta^{(1)}_{k\ell},\ \beta^{(1)}_k,\ \zeta^{(2)}_{k\ell},\ \beta^{(2)}_k,\ \theta_{,k})\,,\\
q_i &= q_i(d_{k\ell},\ \zeta^{(1)}_{k\ell},\ \beta^{(1)}_k,\ \zeta^{(2)}_{k\ell},\ \beta^{(2)}_k,\ \theta_{,k})\,,
\end{aligned} \tag{5.19}$$

but since the nondissipative constitutive equations (5.13) - (5.15) depend on the $y_{i,M}$, $w^{(1)}_i$, $w^{(1)}_{i,M}$, $w^{(2)}_i$, $w^{(2)}_{i,M}$ and θ, there is no logical reason to exclude them from the dissipative constitutive equations[33]. Hence, on account of the chain rule of differentiation, we may write

$$\begin{aligned}
{}^D\tau^S_{ij} &= {}^D\tau^S_{ij}(d_{k\ell},\ \zeta^{(1)}_{k\ell},\ \beta^{(1)}_k,\ \zeta^{(2)}_{k1},\ \beta^{(2)}_k,\ \theta_{,M},\ y_{k,M},\ w^{(1)}_k,\ w^{(1)}_{k,M},\ w^{(2)}_k,\ w^{(2)}_{k,M},\ \theta)\,,\\
{}^D D^{(1)}_{ij} &= {}^D D^{(1)}_{ij}(d_{k\ell},\ \zeta^{(1)}_{k\ell},\ \beta^{(1)}_k,\ \zeta^{(2)}_{k\ell},\ \beta^{(2)}_k,\ \theta_{,M},\ y_{k,M},\ w^{(1)}_k,\ w^{(1)}_{k,M},\ w^{(2)}_k,\ w^{(2)}_{k,M},\ \theta)\,,\\
{}^D \mathcal{J}^{(1)}_{j} &= {}^D \mathcal{J}^{(1)}_{j}(d_{k\ell},\ \zeta^{(1)}_{k\ell},\ \beta^{(1)}_k,\ \zeta^{(2)}_{k\ell},\ \beta^{(2)}_k,\ \theta_{,M},\ y_{k,M},\ w^{(1)}_k,\ w^{(1)}_{k,M},\ w^{(2)}_k,\ w^{(2)}_{k,M},\ \theta)\,,\\
{}^D D^{(2)}_{ij} &= {}^D D^{(2)}_{ij}(d_{k\ell},\ \zeta^{(1)}_{k\ell},\ \beta^{(1)}_k,\ \zeta^{(2)}_{k\ell},\ \beta^{(2)}_k,\ \theta_{,M},\ y_{k,M},\ w^{(1)}_k,\ w^{(1)}_{k,M},\ w^{(2)}_k,\ w^{(2)}_{k,M},\ \theta)\,,\\
{}^D \mathcal{J}^{(2)}_{j} &= {}^D \mathcal{J}^{(2)}_{j}(d_{k\ell},\ \zeta^{(1)}_{k\ell},\ \beta^{(1)}_k,\ \zeta^{(2)}_{k\ell},\ \beta^{(2)}_k,\ \theta_{,M},\ y_{k,M},\ w^{(1)}_k,\ w^{(1)}_{k,M},\ w^{(2)}_k,\ w^{(2)}_{k,M},\ \theta)\,,\\
q_i &= q_i(d_{k\ell},\ \zeta^{(1)}_{k\ell},\ \beta^{(1)}_k,\ \zeta^{(2)}_{k\ell},\ \beta^{(2)}_k,\ \theta_{,M},\ y_{k,M},\ w^{(1)}_k,\ w^{(1)}_{k,M},\ w^{(2)}_k,\ w^{(2)}_{k,M},\ \theta)\,,
\end{aligned} \tag{5.20}$$

for the general functional dependence of the dissipative constitutive equations. Now, in order that the dissipative portions of the constitutive equations satisfy the principle of material objectivity[13,14], all variables in (5.20) must be objective, i.e., they must transform as tensors under time-dependent proper orthogonal transformations. All variables in (5.20) save $d_{k\ell}$, $\zeta^{(1)}_{k\ell}$, $\beta^{(1)}_k$, $\zeta^{(2)}_{k\ell}$ and $\beta^{(2)}_k$ satisfy this latter requirement trivially, since they are not time-differentiated quantities, and $d_{k\ell}$ is known[34] to be objective and $\beta^{(1)}_k$, $\beta^{(2)}_k$, $\zeta^{(1)}_{k\ell}$ and $\zeta^{(2)}_{k\ell}$ may readily be shown to be objective vectors and tensors, respectively. To see this consider

$$y^*_k + w^{(1)*}_k = Q_{k\ell}(t)(y_\ell + w^{(1)}_\ell) + b_k(t) , \quad y^*_k = Q_{k\ell}(t)y_\ell + b_k(t) , \tag{5.21}$$

where $Q_{k\ell}(t)$ represents an arbitrary time-dependent proper orthogonal transformation and $b_k(t)$ an arbitrary time-dependent translation. In (5.21) the starred quantities represent either the motion as seen from an orthogonal coordinate system in arbitrary rigid motion with respect to ours or the motion plus a superposed rigid motion as seen from our coordinate system. From (5.21), we obtain

$$w^{(1)*}_k = Q_{k\ell}(t)w^{(1)}_\ell , \tag{5.22}$$

the material time derivative of which yields

$$dw^{(1)*}_k/dt = Q_{k\ell}dw^{(1)}_\ell/dt + (dQ_{k\ell}/dt)w^{(1)}_\ell . \tag{5.23}$$

Taking the spatial gradient of (5.23) and employing $(5.21)_2$, we find

$$\partial(dw^{(1)*}_k/dt)\partial y^*_m = Q_{mr}Q_{k\ell}\partial(dw^{(1)}_\ell/dt)/\partial y_r + Q_{mr}(dQ_{k\ell}/dt)\partial w^{(1)}_\ell/\partial y_r . \tag{5.24}$$

Now, employing the well-known relation[35]

$$dQ_{k\ell}/dt = Q_{i\ell}\overset{*}{\omega}_{ik} - Q_{ki}\omega_{\ell i} , \tag{5.25}$$

in (5.23) and (5.24), respectively, and using (5.22), we obtain

$$dw_k^{(1)*}/dt - w_i^{(1)*}\omega_{ik}^* = Q_{k\ell}(dw_\ell^{(1)*}/dt - w_i^{(1)}\omega_{i\ell}) ,$$

$$\frac{\partial}{\partial y_m^*}(dw_k^{(1)*}/dt) - \frac{\partial w_i^{(1)*}}{\partial y_m^*}\omega_{ik}^* = Q_{k\ell}Q_{mr}\left(\frac{\partial}{\partial y_r}(dw_\ell^{(1)}/dt) - \frac{\partial w_i^{(1)}}{\partial y_r}\omega_{i\ell}\right), \tag{5.26}$$

which shows that $\beta_j^{(1)}$ and $\zeta_{ij}^{(1)}$ are an objective vector and tensor, respectively. Obviously, in the same way we may readily show that $\beta_j^{(2)}$ and $\zeta_{ij}^{(2)}$ constitute an objective vector and tensor, respectively. Now, the quantities on the left-hand sides of (5.20) cannot be arbitrary functions of the variables shown because arbitrary functions of the variables shown will not satisfy the principle of material objectivity[13,14], which requires the constitutive equations to transform appropriately under proper orthogonal transformations. However, if ${}^D T_{ij}^S$, ${}^D D_{ij}^{(1)}$, ${}^D \mathcal{J}_j^{(1)}$, ${}^D D_{ij}^{(2)}$, ${}^D \mathcal{J}_j^{(2)}$ and q_i are expressed in the forms

$${}^D T_{ij}^S = y_{i,K}y_{j,L}T_{KL} , \quad {}^D D_{ij}^{(1)} = y_{i,K}y_{j,L}\Delta_{KL}^{(1)} , \quad {}^D \mathcal{J}_j^{(1)} = y_{j,K}\Phi_K^{(1)} ,$$

$${}^D D_{ij}^{(2)} = y_{i,K}y_{j,L}\Delta_{KL}^{(2)} , \quad {}^D \mathcal{J}_j^{(2)} = y_{j,K}\Phi_K^{(2)} , \quad q_i = y_{i,K}L_K , \tag{5.27}$$

where T_{KL}, $\Delta_{KL}^{(1)}$, $\Phi_K^{(1)}$, $\Delta_{KL}^{(2)}$, $\Phi_K^{(2)}$ and L_K are functions of the variables shown on the respective right-hand sides of (5.20), it may readily be shown using established methods[36] that the principle of material objectivity is satisfied if $\Phi_K^{(1)}$, $\Phi_K^{(2)}$ and L_K are vector invariants in a rigid motion and T_{KL}, $\Delta_{KL}^{(1)}$ and $\Delta_{KL}^{(2)}$ are tensor invariants in a rigid motion. Then the theory[37] of invariant functions of vectors and second rank tensors shows[38] that the required invariance is assured if T_{KL}, $\Delta_{KL}^{(1)}$, $\Phi_K^{(1)}$, $\Delta_{KL}^{(2)}$, $\Phi_K^{(2)}$ and L_K, respectively, are of the form

$$T_{KL} = T_{KL}(R_{MN}, Z_{MN}^{(1)}, B_M^{(1)}, Z_{MN}^{(2)}, B_M^{(2)}, G_M, E_{MN}, N_M^{(1)}P_{MN}^{(1)}, N_M^{(2)}P_{MN}^{(2)}, \theta) ,$$

$$\Delta_{KL}^{(1)} = \Delta_{KL}^{(1)}(R_{MN}, Z_{MN}^{(1)}, B_M^{(1)}, Z_{MN}^{(2)}, B_M^{(2)}, G_M, E_{MN}, N_M^{(1)}, P_{MN}^{(1)}, N_M^{(2)}, P_{MN}^{(2)}, \theta) ,$$

$$\Phi_K^{(1)} = \Phi_K^{(1)}(R_{MN}, Z_{MN}^{(1)}, B_M^{(1)}, Z_{MN}^{(2)}, B_M^{(2)}, G_M, E_{MN}, N_M^{(1)}, P_{MN}^{(1)}, N_M^{(2)}, P_{MN}^{(2)}, \theta) ,$$

$$\Delta_{KL}^{(2)} = \Delta_{KL}^{(2)}(R_{MN}, Z_{MN}^{(1)}, B_M^{(1)}, Z_{MN}^{(2)}, B_M^{(2)}, G_M, E_{MN}, N_M^{(1)}, P_{MN}^{(1)}, N_M^{(2)}, P_{MN}^{(2)}, \theta),$$

$$\Phi_K^{(2)} = \Phi_K^{(2)}(R_{MN}, Z_{MN}^{(1)}, B_M^{(1)}, Z_{MN}^{(2)}, B_M^{(2)}, G_M, E_{MN}, N_M^{(1)}, P_{MN}^{(1)}, N_M^{(2)}, P_{MN}^{(2)}, \theta),$$

$$L_K = L_K(R_{MN}, Z_{MN}^{(1)}, B_M^{(1)}, Z_{MN}^{(2)}, B_M^{(2)}, G_M, E_{MN}, N_M^{(1)}, P_{MN}^{(1)}, N_M^{(2)}, P_{MN}^{(2)}, \theta), \tag{5.28}$$

where

$$R_{MN} = y_{i,M}y_{j,N}d_{ij} = dE_{MN}/dt,\ Z_{MN}^{(1)} = y_{i,M}y_{j,N}\zeta_{ij}^{(1)},\ B_M^{(1)} = y_{i,M}\beta_i^{(1)},$$
$$Z_{MN}^{(2)} = y_{i,M}y_{j,N}\zeta_{ij}^{(2)},\ B_M^{(2)} = y_{i,M}\beta_i^{(2)},\ G_M = \theta_{,M}, \tag{5.29}$$

and E_{MN}, $N_M^{(1)}$, $P_{MN}^{(1)}$, $N_M^{(2)}$ and $P_{MN}^{(2)}$ are defined in (5.12) and (5.10). Now, it must be remembered that although the dependence of T_{KL}, $\Delta_{KL}^{(1)}$, $\Phi_K^{(1)}$, $\Delta_{KL}^{(2)}$, $\Phi_K^{(2)}$ and L_K on E_{MN}, $N_M^{(1)}$, $P_{MN}^{(1)}$, $N_M^{(2)}$ and $P_{MN}^{(2)}$ is arbitrary, there are conditions on their dependence on R_{MN}, $Z_{MN}^{(1)}$, $B_M^{(1)}$, $Z_{MN}^{(2)}$, $B_M^{(2)}$ and G_M on account of the Clausius-Duhem inequality (4.17). Thus the dissipative constitutive equations in the general case are given by (5.27), with (5.28).

Equations (4.11), $(4.8)_{2-5}$, (5.13) - (5.15), (5.27) and (5.28) determine the constitutive equations for our combined continuum. Thus, all that remains in the determination of explicit constitutive equations is the selection of specific forms for ψ, T_{KL}, $\Delta_{KL}^{(1)}$, $\Phi_K^{(1)}$, $\Delta_{KL}^{(2)}$, $\Phi_K^{(2)}$ and L_K. Once the constitutive equations have been determined, we have a determinate theory, which by appropriate substitution can readily be reduced to 10 equations in the 10 dependent variables y_j, $w_j^{(1)}$, $w_j^{(2)}$ and θ. The 10 equations are the three each of (3.16), (3.20), (3.25) and (4.16). In order to have a complete field theory, the boundary (or jump) conditions at moving surfaces of discontinuity have to be adjoined to the aforementioned system of equations. These boundary conditions are determined in the next section.

Before ending this section it is perhaps worth noting the physical fact that the objective tensors $\beta_j^{(1)}$, $\zeta_{ij}^{(1)}$, $\beta_j^{(2)}$ and $\zeta_{ij}^{(2)}$ are nothing more than the portions of $dw_j^{(1)}/dt$, $(dw_j^{(1)}/dt)_{,i}$, $dw_j^{(2)}/dt$ and $(dw_j^{(2)}/dt)_{,i}$, respectively, beyond that of each due to the local rigid body rate of rotation ω_{ij}. As a consequence, if the aforementioned vectors $w_j^{(1)}$ and $w_j^{(2)}$ and tensors $w_{j,i}^{(1)}$ and $w_{j,i}^{(2)}$ are rigidly fixed in the continuum, i.e., with respect to $\boldsymbol{y}$, then the attendant attenuation will vanish.

6. Boundary Conditions

In this section we determine the boundary conditions which must be adjoined to the system of differential equations, as noted in Section 5, in order to formulate boundary value problems. As usual, these boundary (or jump) conditions are determined by applying the integral forms of the pertinent field equations to appropriate limiting regions surrounding the moving (not necessarily material) surface of discontinuity[39] with normal velocity u_n, and assuming that certain variables remain bounded. The pertinent integral forms are (3.2), the integral forms of (3.16), (3.20), (3.25) and (4.17), which take the respective forms

$$\int_S \boldsymbol{n} \cdot \boldsymbol{\tau}\, dS + \int_V \rho \boldsymbol{f}\, dV = \frac{d}{dt} \int_V \rho \boldsymbol{v}\, dV \tag{6.1}$$

$$\int_S n_i D_{ij}^{(1)}\, dS + \int_V \rho^{(1)} f_j^{(1)}\, dV + \int_V \mathfrak{F}_j^{(1)}\, dV = \frac{d}{dt} \int_V \rho^{(1)} \frac{d\eta_j^{(1)}}{dt}\, dV\,, \tag{6.2}$$

$$\int_S n_i D_{ij}^{(2)}\, dS + \int_V \rho^{(2)} f_j^{(2)}\, dV + \int_V \mathfrak{F}_j^{(2)}\, dV = \frac{d}{dt} \int_V \rho^{(2)} \frac{d\eta_j^{(2)}}{dt}\, dV\,, \tag{6.3}$$

$$\frac{d}{dt} \int_V \rho\eta\, dV + \int_S \frac{n_i q_i}{\theta}\, dS = \int_V \rho\Gamma\, dV \geq 0\,, \tag{6.4}$$

where $\rho\Gamma$ is defined in (4.17).

For all the integral forms considered, a volumetric region is taken in the usual way[39], and it is assumed that all pertinent variables remain bounded. The jump conditions obtained from the respective integral forms consisting of (3.2) and (6.1) - (6.4) are

$$u_n[\rho] - n_i[v_i\rho] = 0 \tag{6.5}$$

$$n_i[\tau_{ij}] + u_n[\rho v_j] - n_i[v_i\rho v_j] = 0 \tag{6.6}$$

$$n_i[D_{ij}^{(1)}] + u_n\left[\rho^{(1)}\frac{d\eta_j^{(1)}}{dt}\right] - n_i\left[v_i\,\rho^{(1)}\frac{d\eta_j^{(1)}}{dt}\right] = 0\,, \tag{6.7}$$

$$n_i[D_{ij}^{(2)}] + u_n\left[\rho^{(2)}\frac{d\eta_j^{(2)}}{dt}\right] - n_i\left[v_i\,\rho^{(2)}\frac{d\eta_j^{(2)}}{dt}\right] = 0\,, \tag{6.8}$$

$$n_i[q_i/\theta] - u_n[\rho\eta] + n_i[v_i\rho\eta] \geq 0\,, \tag{6.9}$$

where we have introduced the conventional notation $[C_i]$ for $C_i^+ - C_i^-$ and n_i denotes the components of the unit normal directed from the - to + side of the surface of discontinuity. If the surface of discontinuity is material

$$u_n = n_i v_i^+ = n_i v_i^-\,, \tag{6.10}$$

then (6.5) evaporates and (6.6) - (6.9), respectively, reduce to

$$n_i[\tau_{ij}] = 0\,, \tag{6.11}$$

$$n_i[D_{ij}^{(1)}] = 0\,, \tag{6.12}$$

$$n_i[D_{ij}^{(2)}] = 0\,, \tag{6.13}$$

$$n_i[q_i/\theta] = 0\,. \tag{6.14}$$

Moreover, if θ is continuous, i.e.,

$$[\theta] = 0\,, \tag{6.15}$$

across the surface of discontinuity, Γ is bounded and, from (4.17), in place of (6.14), we have

$$n_i[q_i] = 0 . \tag{6.16}$$

This latter situation, consisting of the jump conditions (6.11) - (6.13), and (6.15) - (6.16), is the most common, and if the body does not abut another solid body but abuts, say, air instead, the boundary conditions are fully defined by the noted equations. However, if a body does abut another solid body and the full field equations have to be satisfied in each region, additional conditions on $[\underset{\sim}{y}]$, $[\underset{\sim}{\eta}^{(1)}]$ and $[\underset{\sim}{\eta}^{(2)}]$ have to be satisfied at the surface of discontinuity. The conditions are usually

$$[\underset{\sim}{y}] = 0 , \quad [\underset{\sim}{\eta}^{(1)}] = 0 , \quad [\underset{\sim}{\eta}^{(2)}] = 0 , \tag{6.17}$$

the latter two of which may, by virtue of (3.24) and $(3.26)_4$, respectively, be written in the form

$$[(1 + r^{(1)})\underset{\sim}{w}^{(1)} + r^{(2)}\underset{\sim}{w}^{(2)}] = 0 , \quad [(1 + r^{(2)})\underset{\sim}{w}^{(2)} + r^{(1)}\underset{\sim}{w}^{(1)}] = 0 . \tag{6.18}$$

Frequently, the thermal conditions are such that we may eliminate either (6.15) or (6.16). Clearly, all boundary expressions, which are not prescribed, may be expressed in terms of the same 10 field variables as the 10 equations mentioned at the end of Section 5 by making the appropriate straight-forward substitutions.

We can determine an energetic jump condition from (4.4), which, although not needed in the solution of many types of boundary-value problem, can be useful for obtaining certain types of information. This jump condition is obtained by applying (4.4) to the aforementioned volumetric region surrounding the (not necessarily material) surface of discontinuity and assuming that all pertinent variables remain bounded, with the result

$$n_j\left[\tau_{jk}v_k + D^{(1)}_{jk}\frac{dw^{(1)}_k}{dt} + D^{(2)}_{jk}\frac{dw^{(2)}_k}{dt} - q_i\right] + u_n[T + \rho\epsilon] - n_j[v_j(T + \rho\epsilon)] = 0 , \tag{6.19}$$

where T is given in (4.3). If the surface is material we have (6.10), and (6.19) reduces to

$$n_j\left[\tau_{jk}v_k + D^{(1)}_{jk}\frac{dw^{(1)}_k}{dt} + D^{(2)}_{jk}\frac{dw^{(2)}_k}{dt} - q_j\right] = 0\,. \qquad (6.20)$$

7. Generalization to N-Constituents

In this section we generalize the equations which have been derived for the three-constituent composite material to a composite with N-constituents. Since the form in which the equations for the three constituent composite material have been written makes the form of the equations for the N-constituent composite rather obvious, we briefly present the basic and essential resulting equations here for completeness without presenting any of the intermediate equations. In fact, where possible we simply refer to the generalization of existing equations without writing new ones.

The equations of Section 2 remain unchanged except for (2.3), (2.9) and (2.11) - (2.13), which, with the exception of the intermediate equation (2.11), take the resulting forms

$$\rho_o = \sum_{m=1}^{N}\rho_o^{(m)}\,, \quad \rho = \sum_{m=1}^{N}\rho^{(m)}\,, \quad \sum_{m=1}^{N}\rho^{(m)}w^{(m)} = 0\,, \quad \sum_{m=1}^{N}\rho^{(m)}\,dw^{(m)}/dt = 0\,. \qquad (7.1)$$

The equations of the conservation of linear momentum for each constituent, i.e., (3.10) - (3.12), remain the same except that the sums of the internal interactions ${}^{L}F^{nm}$ in each equation increase in number to (N-1), where the meaning of ${}^{L}F^{nm}$ is obvious from the discussion in the last paragraph of Section 2 and, of course, the number of such equations increases to N. The equation of the conservation of linear momentum for the combined continuum, (3.16), remains the same and the difference equations of linear momentum, (3.20) and (3.25), take the form

$$D^{(n)}_{ij,i} + \rho^{(n)}\tilde{f}^{(n)}_j - \rho^{(n)}\frac{d^2\eta^{(n)}_j}{dt^2} + \mathfrak{F}^{(n)}_j = 0\,, \quad n = 1,2,\ldots\,(N-1)\,, \qquad (7.2)$$

where

$$\mathfrak{F}_j^{(n)} = \sum_{m\neq n}^{N} {}^{L}F_j^{nm} - r^{(n)} \sum_{m\neq N}^{(N-1)} {}^{L}F_j^{Nm}, \quad \mathfrak{f}_j^{(n)} = f_j^{(n)} - f_j^{(N)},$$

$$\eta_j^{(n)} = w_j^{(n)} + \sum_{n=1}^{(N-1)} r^{(n)} w_j^{(n)}, \quad r^{(n)} = \rho_o^{(n)}/\rho_o^{(N)} . \qquad (7.3)$$

The equations of the conservation of angular momentum for each constituent, (3.13) - (3.15), remain the same except that the sums of the internal interaction couples ${}^{L}\underset{\sim}{C}^{nm}$ increase in number to (N-1), where the meaning of the ${}^{L}\underset{\sim}{C}^{nm}$ is clear from the discussion in the last paragraph of Section 2 and, of course, the number of such equations increases to N. From Eq.(3.27) it is clear that the equation of the conservation of angular momentum for the combined continuum now takes the form

$$\underset{\sim}{e}_\ell e_{\ell ij}\tau_{ij} + \underset{\sim}{e}_\ell e_{\ell kj} \sum_{m=1}^{(N-1)} (w_{k,i}^{(n)} \; D_{ij}^{(n)} - w_k^{(n)} \mathfrak{F}_j^{(n)}) = 0 . \qquad (7.4)$$

The equation of the conservation of energy (4.1) and the definition of the kinetic energy density (4.2) remain the same except that the sums increase to N. Equations (4.3) and (4.4) retain the same form, but the sums increase to (N-1) and all possible quadratic mixed products occur for (N-1) terms. The equivalent of Eqs.(4.5) and (4.6) take the form

$$\underset{\sim}{t} = \sum_{n=1}^{N} \underset{\sim}{t}^{(n)}, \quad \underset{\sim}{t} = \underset{\sim}{n} \cdot \underset{\sim}{\tau}, \qquad (7.5)$$

$$\underset{\sim}{d}^{(n)} = \underset{\sim}{t}^{(n)} - r^{(n)} \underset{\sim}{t}^{(N)}, \quad \underset{\sim}{d}^{(n)} = \underset{\sim}{n} \cdot \underset{\sim}{D}^{(n)}, \quad n = 1,2, \; .... \; (N-1). \quad 7.6)$$

Equations (4.8), (4.9), (4.11), (4.12) and (4.14) remain unchanged. The number of $\underset{\sim}{D}^{(n)}$ and ${}^{L}\underset{\sim}{\mathfrak{F}}^{(n)}$ in Eqs.(4.8) increase to (N-1). The sums in the remaining equations in Section 4, i.e., (4.7), (4.10), (4.13) and (4.15) - (4.17) increase to (N-1). The number of $w_j^{(n)}$ and $w_{j,i}^{(n)}$ occurring in the generalization of the equations in Section 5 is (N-1). Consequently, the generalization of all

equations in Section 5 is obvious, i.e., wherever the $w_j^{(n)}$ occur, there are (N-1) of them in place of 2, including all conjugate quantities. Accordingly, the definitions occurring in (5.10), (5.13) and (5.29) must be increased thus

$$P_{LM}^{(n)} = y_{k,L}w_{k,M}^{(n)}, \quad N_L^{(n)} = y_{k,L}w_k^{(n)}, \quad n=1,2, \ldots (N-1), \tag{7.7}$$

$$\zeta_{kj}^{(n)} = (dw_j^{(n)}/dt)_{,k} - w_{i,k}^{(n)}\omega_{ij}, \quad \beta_j^{(n)} = dw_j^{(n)}/dt - w_i^{(n)}\omega_{ij}, \quad n=1,2, \ldots (N-1) \tag{7.8}$$

$$Z_{MN}^{(n)} = y_{i,M}y_{j,N}\zeta_{ij}^{(n)}, \quad B_M^{(n)} = y_{i,M}\beta_i^{(n)}, \quad n=1,2, \ldots (N-1), \tag{7.9}$$

and the constitutive relations occurring in (5.14), (5.15) and (5.27) must be increased thus

$$^R\mathcal{D}_{ij}^{(n)} = \rho y_{i,M}y_{j,L}\,\partial\psi/\partial P_{LM}^{(n)}, \quad ^R\mathcal{J}_j^{(n)} = -\rho y_{i,L}\,\partial\psi/\partial N_L^{(n)}, \tag{7.10}$$

$$^D\mathcal{D}_{ij}^{(n)} = y_{i,K}y_{j,L}\Delta_{KL}^{(n)}, \quad ^D\mathcal{J}_j^{(n)} = y_{i,K}\Phi_K^{(n)}, \quad n=1,2, \ldots (N-1), \tag{7.11}$$

and the associated dependence in (5.11) and (5.28) must be increased to suit, i.e., must contain the variables with superscripts from 1 to (N-1).

The jump conditions (6.5), (6.6), (6.9), (6.11), (6.14) - (6.16) and (6.17), remain unchanged and the sums in (6.19) and (6.20) increase to (N-1). The remaining jump conditions, i.e., (6.7), (6.8), (6.12), (6.13), $(6.17)_{2-3}$ and (6.18) take the respective forms

$$n_{\underset{\sim}{i}}[\mathcal{D}_{ij}^{(n)}\underset{\sim}{]} + u_{\underset{\sim}{n}}[\rho^{(n)}d\eta_j^{(n)}/dt\underset{\sim}{]} - n_{\underset{\sim}{i}}[v_i\rho^{(n)}d\eta_j^{(n)}/dt\underset{\sim}{]} = 0, \tag{7.12}$$

$$n_{\underset{\sim}{i}}[\mathcal{D}_{ij}^{(n)}\underset{\sim}{]} = 0, \tag{7.13}$$

$$\underset{\sim}{[}\eta_j^{(n)}\underset{\sim}{]} = 0, \tag{7.14}$$

$$\underset{\sim}{[}w_j^{(n)} + \sum_{m=1}^{N-1} r^{(m)}w_j^{(m)}\underset{\sim}{]} = 0. \tag{7.15}$$

8. Piola-Kirchhoff Form of the Equations

Up to this point all the equations have been written in terms of present (or spatial) coordinates. Since the reference (or material) coordinates of material points are known while the present (or spatial) coordinates are not, it is advantageous to have the equations written in terms of the reference coordinates. To this end, analogous to the Piola-Kirchhoff stress tensor K_{Lj}, which is defined by

$$n_i \tau_{ij}\, dS = N_L K_{Lj}\, dS_o\,, \tag{8.1}$$

we define the reference relative stress tensors $\mathcal{D}^{(n)}_{Lj}$ by

$$n_i D^{(n)}_{ij}\, dS = N_L \mathcal{D}^{(n)}_{Lj}\, dS_o\,, \tag{8.2}$$

where dS_o and N_L denote the magnitude of and unit normal to an element of area in the reference configuration, which has magnitude dS and unit normal n_i in the present configuration. By virtue of the well-known relation[40]

$$n_i\, dS = J X_{L,i} N_L\, dS_o\,, \tag{8.3}$$

from (8.1) and (8.2), in the usual way, we find

$$K_{Lj} = J X_{L,i} \tau_{ij}\,, \quad \mathcal{D}^{(n)}_{Lj} = J X_{L,i} D^{(n)}_{ij}\,. \tag{8.4}$$

Now, using $(8.4)_1$ and $(8.4)_2$, respectively, with (3.16) and (7.2) and employing (2.8). (2.10) and the well-known identity[41]

$$(J X_{R,i})_{,R} = 0\,, \tag{8.5}$$

we obtain, respectively,

$$K_{Lj,L} + \rho_o f_j = \rho_o\, dv_j/dt\,, \tag{8.6}$$

$$\mathcal{D}^{(n)}_{Lj,L} + \rho_o^{(n)} \tilde{f}^{(n)}_j + J\, \mathfrak{F}^{(n)}_j = \rho_o^{(n)} d^2 \eta^{(n)}_j / dt^2\,, \quad n = 1, 2, \ldots\ (N-1)\,, \tag{8.7}$$

the first of which is the Piola-Kirchhoff form of the stress equations of motion and the second of which is the reference form of the (N-1) relative stress equations of motion.

Analogous to the foregoing, we now define the reference heat flux vector Q_L by

$$n_i q_i \, dS = N_L Q_L \, dS_o , \tag{8.8}$$

which with (8.3) yields

$$Q_L = J X_{L,i} q_i . \tag{8.9}$$

Substituting from (8.9), (5.27), (5.29) and (7.8) - (7.10) into the generalization of (4.16) for N-constituents and employing (8.5) and (2.10), we obtain

$$J\left[T_{KL} dE_{KL}/dt + \sum_{m=1}^{(N-1)} (\Delta_{KL}^{(m)} Z_{KL}^{(m)} - \Phi_K^{(m)} B_K^{(m)})\right] - Q_{L,L} = \rho_o \theta \, d\eta/dt , \tag{8.10}$$

which is the reference form of the dissipation equation.

In view of (8.4), (8.9), (5.13) - (5.15), (2.10), (5.27), (7.10) and (7.11), the pertinent constitutive equations for this section may be written in the form

$$^{R}K_{Lj} = \rho_o y_{j,M} \frac{\partial \psi}{\partial E_{LM}} + \rho_o \sum_{m=1}^{(N-1)} \left[\frac{\partial \psi}{\partial N_L^{(m)}} w_j^{(m)} + \frac{\partial \psi}{\partial P_{LM}^{(m)}} w_{j,M}^{(m)}\right] ,$$

$$J\,{}^{R}\mathcal{J}_j^{(n)} = - \rho_o y_{j,L} \frac{\partial \psi}{\partial N_L^{(n)}} , \quad {}^{R}\mathcal{B}_{Lj}^{(n)} = \rho_o y_{j,K} \frac{\partial \psi}{\partial P_{KL}^{(n)}} , \tag{8.11}$$

$$^{D}K_{Lj} = J y_{j,M} T_{LM} , \quad J\,{}^{D}\mathcal{J}_j^{(n)} = J y_{j,K} \Phi_K^{(n)} ,$$

$$^{D}\mathcal{B}_{Lj}^{(n)} = J y_{j,M} \Delta_{LM}^{(n)} , \quad Q_K = J L_K , \tag{8.12}$$

where

$$K_{Lj} = {}^{R}K_{Lj} + {}^{D}K_{Lj} , \quad \mathcal{J}_j^{(n)} = {}^{R}\mathcal{J}_j^{(n)} + {}^{D}\mathcal{J}_j^{(n)} , \quad \mathcal{B}_{Lj}^{(n)} = {}^{R}\mathcal{B}_{Lj}^{(n)} + {}^{D}\mathcal{B}_{Lj}^{(n)} , \tag{8.13}$$

and ψ, T_{LM}, $\phi_K^{(n)}$, $\Delta_{LM}^{(n)}$ and L_K are the generalized versions of (5.11) and (5.28), which are discussed in Section 7. In view of (8.1), (8.2) and (8.6) - (8.8), in reference coordinates the boundary conditions (6.6), (7.12), (6.9), (6.11), (7.13), (6.14) and (6.16) take the respective forms

$$N_L[K_{Lj}] + U_N\rho_o[v_j] = 0\,, \tag{8.14}$$

$$N_L[\mathcal{B}_{Lj}^{(n)}] + U_N\rho_o[d\eta_j^{(n)}/dt] = 0\,, \quad n = 1,2, \;.... \;(N-1)\,, \tag{8.15}$$

$$N_L[Q_L/\theta] - U_N\rho_o[\eta] = 0\,, \tag{8.16}$$

$$N_L[K_{Lj}] = 0\,, \quad N_L[\mathcal{B}_{Lj}^{(n)}] = 0\,, \quad n = 1,2, \;.... \;(N-1)\,, \tag{8.17}$$

$$N_L[Q_L] \geq 0\,, \quad N_L[Q_L] = 0\,, \tag{8.18}$$

where U_N is the intrinsic velocity[42] of the singular surface, i.e., the velocity of the singular surface in the reference coordinate system. The boundary conditions (6.15), $(6.17)_1$, (7.14) and (7.15) remain unchanged, while (6.5) degenerates to nothing in the reference coordinate description.

9. Linear Equations for the Two-Constituent Composite

In this section we obtain the linear equations for the two-constituent composite material in the absence of dissipation from the general nonlinear equations for the N-constituent composite material presented in Sections 7 and 8. To this end we first note that for the two-constituent composite $N = 2$. Then, in the usual way, we define the mechanical displacement vector u_M by

$$y_i = \delta_{iM}(X_M + u_M)\,, \tag{9.1}$$

where δ_{iM} is a translation operator, which serves to translate a vector from the present to the reference position and vice-versa and is required for notational consistency and clarity because of the use of capital and lower case

indices, respectively, to refer to the reference and present coordinates of the center of mass of material points. From (9.1), we have

$$y_{i,L} = \delta_{iL} + \delta_{iM} u_{M,L}, \tag{9.2}$$

and substituting from (9.2) into (5.12) and neglecting products of $u_{M,L}$, we obtain

$$E_{LM} \approx \epsilon_{LM} = \frac{1}{2}\,(u_{M,L} + u_{L,M}), \tag{9.3}$$

which is the usual infinitesimal strain tensor. Similarly, substituting from (9.2) into (7.7) for N = 2, we obtain

$$P^{(1)}_{LM} = \delta_{kL} w^{(1)}_{k,M} = w^{(1)}_{L,M}, \quad N^{(1)}_{L} = \delta_{kL} w^{(1)}_{k} = w^{(1)}_{L}, \tag{9.4}$$

where we have taken the liberty of utilizing capital indices to denote the Cartesian components of the relative displacement vector $\underset{\sim}{w}^{(1)}$ in the linear description being obtained here. Since $R_{K_{Lj}}$, $R_{\mathcal{J}_j}^{(1)}$ and $R_{\mathcal{J}_{Lj}}^{(1)}$ are assumed to vanish when u_M and $w^{(1)}_L$ vanish, in this linear theory ψ must be a homogeneous quadratic function of the form

$$\rho_o\psi = \frac{1}{2}\, c_{KLMN}\epsilon_{KL}\epsilon_{MN} + \frac{1}{2}\, a_{KL} w^{(1)}_{K} w^{(1)}_{L} + \frac{1}{2}\, b_{KLMN} w^{(1)}_{K,L} w^{(1)}_{M,N} +$$

$$\alpha_{MKL}\epsilon_{KL} w^{(1)}_{M} + \beta_{KLMN}\epsilon_{KL} w^{(1)}_{M,N} + \gamma_{MKL} w^{(1)}_{K,L} w^{(1)}_{M}, \tag{9.5}$$

where the c_{KLMN} are the usual elastic constants of ordinary linear elasticity, the a_{KL} may be called the difference displacement elastic constants, the b_{KLMN}, the relative elastic constants and α_{MKL}, β_{KLMN} and γ_{MKL}, the respective coupling constants. In the arbitrarily anisotropic case, there are the usual 21 independent c_{KLMN}, 6 independent a_{KL}, 45 independent b_{KLMN}, 18 independent α_{MKL}, 54 independent β_{KLMN} and 27 independent γ_{MKL}, for a total of 171 independent material constants.

Now, substitiţing from $(7.3)_3$, (8.11), (8,13) and (9.2) - (9.5) into (8.6) and (8.7) and neglecting all nonlinear terms, we obtain

$$K_{LM,L} + \rho_o f_M = \rho_o \ddot{u}_M, \tag{9.6}$$

$$\mathscr{B}^{(1)}_{LM,L} + \frac{r^{(1)}}{1 + r^{(1)}} \rho_o \tilde{f}^{(1)}_M + \mathscr{F}^{(1)}_M = r^{(1)} \rho_o \ddot{w}^{(1)}_M , \tag{9.7}$$

where we have employed the relations

$$v_j = \partial y_j/\partial t = \delta_{jM}\partial u_M/\partial t = \delta_{jM}\dot{u}_M, \quad \tilde{f}^{(1)}_j = \delta_{jM}\tilde{f}^{(1)}_M$$

$$K_{Lj} = \delta_{jM}K_{LM}, \quad \mathscr{B}^{(1)}_{Lj} = \delta_{jM}\mathscr{B}^{(1)}_{LM}, \quad f_j = \delta_{jM}f_M, \quad \mathscr{F}^{(1)}_j = \delta_{jM} \mathscr{F}^{(1)}_M, \tag{9.8}$$

and

$$K_{LM} = \frac{\partial(\rho_o\psi)}{\partial\epsilon_{LM}}, \quad \mathscr{B}^{(1)}_{LM} = \frac{\partial(\rho_o\psi)}{\partial w^{(1)}_{M,L}}, \quad \mathscr{F}^{(1)}_M = -\frac{\partial(\rho_o\psi)}{\partial w^{(1)}_M} . \tag{9.9}$$

From (9.5) and (9.9) we have

$$\begin{aligned} K_{LM} &= c_{LMKN}\epsilon_{KN} + \alpha_{KLM}w^{(1)}_K + \beta_{LMKN}w^{(1)}_{K,N} , \\ \mathscr{B}^{(1)}_{LM} &= \beta_{KNML}\epsilon_{KN} + \gamma_{KML}w^{(1)}_K + b_{MLKN}w^{(1)}_{K,N} , \\ \mathscr{F}^{(1)}_M &= -\alpha_{MKL}\epsilon_{KL} - a_{ML}w^{(1)}_L - \gamma_{MKL}w^{(1)}_{K,L} . \end{aligned} \tag{9.10}$$

The substitution of (9.10), with (9.3), into (9.6) and (9.7), respectively, yields

$$c_{LMKN}u_{K,NL} + \alpha_{KLM}w^{(1)}_{K,L} + \beta_{LMKN}w^{(1)}_{K,NL} + \rho_o f_M = \rho_o \ddot{u}_M , \tag{9.11}$$

$$\begin{aligned} &\beta_{KNLM}u_{K,NL} + \gamma_{KLM}w^{(1)}_{K,L} + b_{LMKN}w^{(1)}_{K,NL} + \frac{r^{(1)}}{1 + r^{(1)}} \rho_o \tilde{f}^{(1)}_M \\ &- \alpha_{MKL}u_{K,L} - a_{ML}w^{(1)}_L - \gamma_{MKL}w^{(1)}_{K,L} = r^{(1)}\rho_o \ddot{w}^{(1)}_M , \end{aligned} \tag{9.12}$$

which constitute 6 linear differential equations in the 6 dependent variables u_K and $w^{(1)}_K$. To this system of equations we must adjoin the linear boundary

conditions across material surfaces of discontinuity, which are obtained by substituting from $(9.8)_{3-4}$ into (8.17) for N = 2 with the result,

$$N_{L}[K_{LM}] = 0\,, \quad N_{L}[\mathcal{B}^{(1)}_{LM}] = 0\,. \qquad (9.13)$$

If the surface abuts space, one side of the jump brackets in (9.13) determines applied traction terms in the usual way. On the other hand if the body abuts another solid body, we must obtain the associated jump conditions by substituting from (9.1) and (9.4) into $(6.17)_1$ and (7.15), respectively, with the result

$$[(1 + r^{(1)})w^{(1)}_K] = 0\,. \qquad (9.14)$$

The linear equations for the two constituent composite with discontinuous reinforcement (chopped fiber) can be obtained from the foregoing equations in this section simply by setting b_{KLMN}, β_{KLMN} and γ_{KLM} equal to zero wherever they occur. Under these circumstances $\mathcal{B}^{(1)}_{LM}$ vanishes and the boundary conditions $(9.13)_2$ and (9.14) do not exist.

10. Material Symmetry Considerations

In this section we obtain the linear equations for the isotropic and transversely isotropic two-constituent composite material. Although we can obtain these equations directly from the arbitrarily anisotropic equations presented in Section 9 by writing the tensors for the particular symmetry involved, it is advantageous to return to the stored energy function $\rho_0\psi$ and write all the quadratic scalar invariants first for the isotropic material and then for the transversely isotropic material, especially when a great deal of symmetry exists, as in these two cases. The tables of integrity bases provided by Spencer[43,44] for the two transformation groups involved prove to be extremely valuable in obtaining the independent quadratic invariants. We are concerned with the

quadratic invariants of a symmetric tensor ϵ_{LM}, an asymmetric tensor $w^{(1)}_{L,M}$ and a vector $w^{(1)}_{L}$. Since Spencer systematically considers symmetric tensors, antisymmetric tensors and the skew-symmetric tensors of vectors, we must decompose the asymmetric tensor $w^{(1)}_{L,M}$ into its symmetric and antisymmetric parts p^{S}_{LM} and p^{A}_{LM}, respectively, and write the skew-symmetric tensor $W^{(1)}_{LM}$ of the vector $w^{(1)}_{K}$, thus

$$p^{S}_{LM} = \frac{1}{2}\,(w^{(1)}_{L,M} + w^{(1)}_{M,L})\,, \quad p^{A}_{LM} = \frac{1}{2}\,(w^{(1)}_{L,M} - w^{(1)}_{M,L})\,, \tag{10.1}$$

$$W_{LM} = e_{LMK}w^{(1)}_{K}\ . \tag{10.2}$$

Then $\rho_o\psi$ can be written as a quadratic polynomial invariant in the sum of the invariant products of two symmetric tensors ϵ_{LM} and p^{S}_{LM}, an antisymmetric tensor p^{A}_{LM} and a skew-symmetric tensor W_{LM}.

For isotropic materials possessing a center of symmetry, $\rho_o\psi$ is a scalar invariant under the full orthogonal group. Spencer lists[44] the basic invariants of a number of second order symmetric and antisymmetric tensors for the proper orthogonal group. From this list all quadratic invariants under the full orthogonal group may readily be obtained. Thus we find that for an isotropic material the homogeneous quadratic function $\rho_o\psi$ may be written in the form

$$\begin{aligned}\rho_o\psi = {} & \frac{\lambda}{2}\,\epsilon_{KK}\epsilon_{LL} + \mu\epsilon_{KL}\epsilon_{LK} + \beta_1\epsilon_{KK}p^{S}_{LL} + \beta_2\epsilon_{KL}p^{S}_{LK} + \frac{1}{2}\,b_1p^{S}_{KK}p^{S}_{LL} \\ & + b_2p^{S}_{KL}p^{S}_{LK} + b_3p^{A}_{KL}p^{A}_{LK} + \frac{1}{2}\,a_1w^{(1)}_{K}w^{(1)}_{K}\ .\end{aligned} \tag{10.3}$$

Substituting from (10.3) into (9.9) and employing (9.3) and (10.1), we obtain

$$K_{LM} = \lambda u_{K,K}\delta_{LM} + \mu(u_{L,M} + u_{M,L}) + \beta_1 w^{(1)}_{K,K}\delta_{LM} + \frac{1}{2}\beta_2(w^{(1)}_{L,M} + w^{(1)}_{(M,L)}) ,$$

$$\mathcal{B}^{(1)}_{LM} = \beta_1 u_{K,K}\delta_{LM} + \frac{1}{2}\beta_2(u_{L,M} + u_{M,L}) + b_1 w^{(1)}_{K,K}\delta_{LM} + b_2(w^{(1)}_{L,M} + w^{(1)}_{M,L})$$

$$+ b_3(w^{(1)}_{L,M} - w^{(1)}_{M,L}) ,$$

$$\mathfrak{F}^{(1)}_M = - a_1 w^{(1)}_M , \tag{10.4}$$

which are the linear constitutive equations for the isotropic two-constituent composite material. Substituting from (10.5) into (9.6) and (9.7) and ignoring the external body forces, we obtain

$$(\lambda + \mu)u_{K,KM} + \mu u_{M,KK} + \left(\beta_1 + \frac{1}{2}\beta_2\right)w^{(1)}_{K,KM} + \frac{1}{2}\beta_2 w^{(1)}_{M,KK} = \rho_o \ddot{u}_M , \tag{10.5}$$

$$\left(\beta_1 + \frac{1}{2}\beta_2\right) + u_{K,KM} + \frac{1}{2}\beta_2 u_{M,KK} + (b_1 + b_2 + b_3)w^{(1)}_{K,KM}$$

$$+ (b_2 - b_3)w^{(1)}_{M,KK} - a_1 w^{(1)}_M = r^{(1)}\rho_o \ddot{w}^{(1)}_M , \tag{10.6}$$

which are the equations of motion for the isotropic two constituent composite material.

A material with a single preferred direction which is the same at every point is said to be transversely isotropic. For such a material $\rho_o\psi$ is a scalar invariant under rotations about the preferred direction, which we take along the x_3-axis. The transformations under which we have invariance are the rotations about the x_3-axis, and reflections in the planes containing the x_3-axis. Spencer gives a list[44] of invariants under this transformation. In addition to the foregoing we require invariance under reflections in the plane perpendicular to the x_3-axis. From Spencer's list[44] all quadratic invariants satisfying the latter additional symmetry requirement may readily be obtained.

Thus we find that for our transversely isotropic material the homogeneous quadratic function $\rho_o\psi$ can be written in the form

$$\begin{aligned}\rho_o\psi = {} & \frac{1}{2}\hat{c}_1\epsilon_{\alpha\alpha}\epsilon_{\beta\beta} + \hat{c}_2\epsilon_{\alpha\alpha}\epsilon_{33} + \hat{c}_3\epsilon_{\alpha\beta}\epsilon_{\beta\alpha} + \hat{c}_4\epsilon_{3\alpha}\epsilon_{\alpha 3} + \frac{1}{2}\hat{c}_5\epsilon_{33}\epsilon_{33} \\ & + \hat{\beta}_1\epsilon_{\alpha\alpha}p^S_{\beta\beta} + \hat{\beta}_2\epsilon_{\alpha\beta}p^S_{\beta\alpha} + \hat{\beta}_3\epsilon_{\alpha\alpha}p^S_{33} + \hat{\beta}_4\epsilon_{3\alpha}p^S_{\alpha 3} + \hat{\beta}_5\epsilon_{33}p^S_{\alpha\alpha} \\ & + \hat{\beta}_6\epsilon_{33}p^S_{33} + \hat{b}_1 p^S_{\alpha\beta}p^S_{\beta\alpha} + \frac{1}{2}\hat{b}_2 p^S_{\alpha\alpha}p^S_{\beta\beta} + \hat{b}_3 p^S_{\alpha\alpha}p^S_{33} + \hat{b}_4 p^S_{3\alpha}p^S_{\alpha 3} \\ & + \frac{1}{2}\hat{b}_5 p^S_{33}p^S_{33} + \hat{b}_6 p^A_{\alpha\beta}p^A_{\alpha\beta} + \hat{b}_7 p^A_{3\alpha}p^A_{3\alpha} + \frac{1}{2}\hat{a}_1 w^{(1)}_\alpha w^{(1)}_\alpha + \frac{1}{2}\hat{a}_2 w^{(1)}_3 w^{(1)}_3 ,\end{aligned} \tag{10.7}$$

where the greek indices take the values 1 and 2 and skip 3.

Substituting from (10.7) into (9.9) and employing (9.3) and (10.1), we obtain

$$\begin{aligned}
K_{\alpha\beta} &= \hat{c}_1 u_{\gamma,\gamma}\delta_{\alpha\beta} + \hat{c}_2 u_{3,3}\delta_{\alpha\beta} + \hat{c}_3(u_{\alpha,\beta}+u_{\beta,\alpha}) + \hat{\beta}_1 w^{(1)}_{\gamma,\gamma}\delta_{\alpha\beta} \\
&\quad + \hat{\beta}_3 w^{(1)}_{3,3}\delta_{\alpha\beta} + \frac{1}{2}\hat{\beta}_2(w^{(1)}_{\alpha,\beta} + w^{(1)}_{\beta,\alpha}) , \\
K_{\alpha 3} &= K_{3\alpha} = \hat{c}_4(u_{3,\alpha}+u_{\alpha,3}) + \frac{1}{2}\hat{\beta}_4(w^{(1)}_{3,\alpha} + w^{(1)}_{\alpha,3}) , \\
K_{33} &= \hat{c}_2 u_{\alpha,\alpha} + \hat{c}_5 u_{3,3} + \hat{\beta}_5 w^{(1)}_{\alpha,\alpha} + \hat{\beta}_6 w^{(1)}_{3,3} , \\
\mathcal{B}^{(1)}_{\alpha\beta} &= \hat{\beta}_1 u_{\gamma,\gamma}\delta_{\alpha\beta} + \frac{1}{2}\hat{\beta}_2(u_{\alpha,\beta} + u_{\beta,\alpha}) + \hat{\beta}_5 u_{3,3}\delta_{\alpha\beta} + \hat{b}_1(w^{(1)}_{\alpha,\beta} + w^{(1)}_{\beta,\alpha}) \\
&\quad + \hat{b}_2 w^{(1)}_{\gamma,\gamma}\delta_{\alpha\beta} + \hat{b}_3 w^{(1)}_{3,3}\delta_{\alpha\beta} + \hat{b}_6(w^{(1)}_{\beta,\alpha} - w^{(1)}_{\alpha,\beta}) , \\
\mathcal{B}^{(1)}_{\alpha 3} &= \frac{1}{2}\hat{\beta}_4(u_{3,\alpha} + u_{\alpha,3}) + \hat{b}_4(w^{(1)}_{3,\alpha} + w^{(1)}_{\alpha,3}) + \frac{1}{2}\hat{b}_7(w^{(1)}_{3,\alpha} - w^{(1)}_{\alpha,3}) , \\
\mathcal{B}^{(1)}_{3\alpha} &= \frac{1}{2}\hat{\beta}_4(u_{3,\alpha} + u_{\alpha,3}) + \hat{b}_4(w^{(1)}_{3,\alpha} + w^{(1)}_{\alpha,3}) + \frac{1}{2}\hat{b}_7(w^{(1)}_{\alpha,3} - w^{(1)}_{3,\alpha}) , \\
\mathcal{B}^{(1)}_{33} &= \hat{\beta}_3 u_{\alpha,\alpha} + \hat{\beta}_6 u_{3,3} + \hat{b}_3 w^{(1)}_{\alpha,\alpha} + \hat{b}_5 w^{(1)}_{3,3} , \\
\mathfrak{F}^{(1)}_\beta &= -\hat{a}_1 w^{(1)}_\beta , \\
\mathfrak{F}^{(1)}_3 &= -\hat{a}_2 w^{(1)}_3 ,
\end{aligned} \tag{10.8}$$

which are the linear constitutive equations for the transversely isotropic two-constituent composite material with x_3 the preferred direction. Substituting from (10.8) into (9.6) and (9.7) and ignoring the external body forces, we obtain

$$\hat{c}_1 u_{\alpha,\alpha\beta} + \hat{c}_2 u_{3,3\beta} + \hat{c}_3 (u_{\alpha,\beta\alpha} + u_{\beta,\alpha\alpha}) + \hat{\beta}_1 w^{(1)}_{\alpha,\alpha\beta} + \frac{1}{2}\hat{\beta}_2 (w^{(1)}_{\alpha,\beta\alpha} + w^{(1)}_{\beta,\alpha\alpha})$$
$$+ \hat{\beta}_3 w^{(1)}_{3,3\beta} + \hat{c}_4 (u_{3,\beta 3} + u_{\beta,33}) + \frac{1}{2}\hat{\beta}_4 (w^{(1)}_{3,\beta 3} + w^{(1)}_{\beta,33}) = \rho_o \ddot{u}_\beta ,$$

$$\hat{c}_2 u_{\alpha,\alpha 3} + \hat{c}_4 (u_{3,\alpha\alpha} + u_{\alpha,3\alpha}) + \frac{1}{2}\hat{\beta}_4 (w^{(1)}_{3,\alpha\alpha} + w^{(1)}_{\alpha,3\alpha}) + \hat{c}_5 u_{3,33} + \hat{\beta}_5 w^{(1)}_{\alpha,\alpha 3}$$
$$+ \hat{\beta}_6 w^{(1)}_{3,33} = \rho_o \ddot{u}_3 , \qquad (10.9)$$

$$\hat{\beta}_1 u_{\alpha,\alpha\beta} + \frac{1}{2}\hat{\beta}_2 (u_{\alpha,\alpha\beta} + u_{\beta,\alpha\alpha}) + \frac{1}{2}\hat{\beta}_4 (u_{3,3\beta} + u_{\beta,33}) +$$
$$\hat{\beta}_5 u_{3,3\beta} + \hat{b}_1 (w^{(1)}_{\alpha,\alpha\beta} + w^{(1)}_{\beta,\alpha\alpha}) + \hat{b}_2 w^{(1)}_{\alpha,\alpha\beta} + \hat{b}_3 w^{(1)}_{3,3\beta} + \hat{b}_6 (w^{(1)}_{\beta,\alpha\alpha} - w^{(1)}_{\alpha,\beta\alpha}$$
$$+ \hat{b}_4 (w^{(1)}_{3,\beta 3} + w^{(1)}_{\beta,33}) + \frac{1}{2}\hat{b}_7 (w^{(1)}_{\beta,33} - w^{(1)}_{3,3\beta}) - \hat{a}_1 w^{(1)}_\beta = r^{(1)} \rho_o \ddot{w}^{(1)}_\beta ,$$

$$\hat{\beta}_3 u_{\alpha,\alpha 3} + \frac{1}{2}\hat{\beta}_4 (u_{3,\alpha\alpha} + u_{\alpha,\alpha 3}) + \hat{\beta}_6 u_{3,33} + \hat{b}_3 w^{(1)}_{\alpha,\alpha 3} + \hat{b}_4 (w^{(1)}_{3,\alpha\alpha} + w^{(1)}_{\alpha,\alpha 3})$$
$$+ \frac{1}{2}\hat{b}_7 (w^{(1)}_{3,\alpha\alpha} - w^{(1)}_{\alpha,3\alpha}) + \hat{b}_5 w^{(1)}_{3,33} - \hat{a}_2 w^{(1)}_3 = r^{(1)} \rho_o \ddot{w}^{(1)}_3 , \qquad (10.10)$$

which are the equations of motion for the transversely isotropic two-constituent composite material with x_3 the preferred direction. We have bothered to write the linear constitutive and differential equations for the transversely isotropic two-constituent composite material in complete detail because we deem this to be a particularly important symmetry for fiber reinforced composites.

11. Wave Propagation

The solution for plane wave propagation in the arbitrarily anisotropic infinite medium may readily be obtained by substituting plane waves in the system of linear equations in (9.11) and (9.12). However, the resulting algebra is sufficiently lengthy and cumbersome that it becomes relatively involved to extract useful physical information from the resulting system. Since the isotropic case contains many of the interesting features concerning the propagation of waves in the two-constituent composite and is much less cumbersome than the general anisotropic case because of the considerably smaller number of material constants, we treat plane wave propagation in two-constituent isotropic composites in this section. To this end as a solution of (10.5) and (10.6) consider

$$u_j = A_j e^{i(\xi n_k X_k - \omega t)} , \quad w_j = B_j e^{i(\xi n_k X_k - \omega t)} , \tag{11.1}$$

where n_k is a unit vector denoting the wave normal. The solution (11.1) satisfies (10.5) and (10.6), provided

$$\begin{aligned} &(\mu \xi^2 - \rho_o \omega^2) A_j + (\lambda + \mu) \xi^2 A_k n_k n_j + \frac{1}{2} \beta_2 \xi^2 B_j \\ &\quad + (\beta_1 + \frac{1}{2} \beta_2) \xi^2 B_k n_k n_j = 0 , \\ &\frac{1}{2} \beta_2 \xi^2 A_j + (\beta_1 + \frac{1}{2} \beta_2) \xi^2 A_k n_k n_j + \left[(b_2 - b_3) \xi^2 + a_1 - r_1 \rho_o \omega^2 \right] B_j \\ &\quad + (b_1 + b_2 + b_3) \xi^2 B_k n_k n_j = 0 . \end{aligned} \tag{11.2}$$

At this point it should be noted that in order to secure the positive definiteness of $\rho_o \psi$ in (10.3), we must have the conditions

$$\mu > 0 , \quad 3\lambda + 2\mu > 0 , \quad b_2 > 0 , \quad 3b_1 + b_2 > 0 , \quad b_3 < 0 , \quad a_1 > 0 . \tag{11.3}$$

Equations (11.2) constitute six linear homogeneous algebraic equations in the A_j and B_j, which may also be regarded as two vectorial equations of the form

$\underset{\sim}{g}_1 = 0$ and $\underset{\sim}{g}_2 = 0$. Since the medium is isotropic, a major reduction in the algebra results if we decompose each vector equation along the direction of the unit normal $\underset{\sim}{n}$ and in the plane normal to $\underset{\sim}{n}$. To this end we write $\underset{\sim}{A}$ and $\underset{\sim}{B}$ in the form

$$\underset{\sim}{A} = \underset{\sim}{n} \cdot \underset{\sim}{A}\underset{\sim}{n} - \underset{\sim}{n} \times \underset{\sim}{n} \times \underset{\sim}{A}, \quad \underset{\sim}{B} = \underset{\sim}{n} \cdot \underset{\sim}{B}\underset{\sim}{n} - \underset{\sim}{n} \times \underset{\sim}{n} \times \underset{\sim}{B}, \tag{11.4}$$

and first we write the equations in the normal direction $\underset{\sim}{n}$, to obtain

$$\begin{aligned} &[(\lambda + 2\mu)\xi^2 - \rho_o\omega^2]\underset{\sim}{n} \cdot \underset{\sim}{A} + (\beta_1 + \beta_2)\xi^2\underset{\sim}{n} \cdot \underset{\sim}{B} = 0, \\ &(\beta_1 + \beta_2)\xi^2\underset{\sim}{n} \cdot \underset{\sim}{A} + [(b_1 + 2b_2)\xi^2 + a_1 - r^{(1)}\rho_o\omega^2]\underset{\sim}{n} \cdot \underset{\sim}{B} = 0, \end{aligned} \tag{11.5}$$

which constitute a system of two linear homogeneous algebraic equations in $\underset{\sim}{n} \cdot \underset{\sim}{A}$ and $\underset{\sim}{n} \cdot \underset{\sim}{B}$, thus showing that purely longitudinal waves exist in the isotropic two-constituent composite. For a nontrivial solution, the determinant of the coefficients of $\underset{\sim}{n} \cdot \underset{\sim}{A}$ and $\underset{\sim}{n} \cdot \underset{\sim}{B}$ must vanish, which yields

$$\begin{aligned} &r^{(1)}\rho_o^2\,\omega^4 - \{[(b_1 + 2b_2)\rho_o + (\lambda + 2\mu)\, r^{(1)}\rho_o]\xi^2 + \rho_o a_1\}\omega^2 \\ &+ \{[(\lambda + 2\mu)(b_1 + 2b_2) - (\beta_1 + \beta_2)^2]\xi^2 + (\lambda + 2\mu)a_1\}\xi^2 = 0. \end{aligned} \tag{11.6}$$

Equation (11.6) governs the propagation of longitudinal waves in the isotropic two-constituent composite. From (11.6) it is clear that for a given wavenumber ξ there are two ω^2, and solving for ω^2 and expanding for small ξ, we obtain

$$\omega_1^2 = \frac{1}{\rho_o}(\lambda + 2\mu)\xi^2 + O(\xi^4), \quad \omega_2^2 = \frac{a_1}{r^{(1)}\rho_o} + \frac{(b_1 + 2b_2)}{r^{(1)}\rho_o}\,\xi^2 + O(\xi^4). \tag{11.7}$$

In view of (11.3), we see from (11.7) that both ω_1 and ω_2 are real for real ξ. Moreover, it is clear that on an ω vs ξ diagram there are two branches, one emanating from $\omega = 0$, $\xi = 0$, with positive initial slope $\sqrt{(\lambda + 2\mu)/\rho_o}$, and the other emanating from $\omega = \sqrt{a_1/r^{(1)}\rho_o}$, $\xi = 0$, with zero initial slope and positive curvature $(b_1 + 2b_2)/\sqrt{r^{(1)}\rho_o a_1}$.

Now we write the equations in the plane normal to $\underset{\sim}{n}$ to obtain

$$(\mu\,\xi^2 - \rho_o\omega^2)\underset{\sim}{n}\times\underset{\sim}{n}\times\underset{\sim}{A} + \frac{1}{2}\,\beta_2\xi^2\underset{\sim}{n}\times\underset{\sim}{n}\times\underset{\sim}{B} = 0\,,\quad \frac{1}{2}\,\beta_2\xi^2\underset{\sim}{n}\times\underset{\sim}{n}\times\underset{\sim}{A} + [(b_2 - b_3)\xi^2$$
$$+ a_1 - r^{(1)}\rho_o\omega^2]\underset{\sim}{n}\times\underset{\sim}{n}\times\underset{\sim}{B} = 0\,, \tag{11.8}$$

which constitute a system of two linear homogeneous algebraic equations in the vectors $\underset{\sim}{n}\times\underset{\sim}{n}\times\underset{\sim}{A}$ and $\underset{\sim}{n}\times\underset{\sim}{n}\times\underset{\sim}{B}$, thus showing that purely transverse waves exist in the isotropic medium. For a nontrivial solution, the determinant of the coefficients of $\underset{\sim}{n}\times\underset{\sim}{n}\times\underset{\sim}{A}$ and $\underset{\sim}{n}\times\underset{\sim}{n}\times\underset{\sim}{B}$ must vanish, which yields

$$r^{(1)}\rho_o^2\,\omega^4 - \{[r^{(1)}\rho_o\mu + \rho_o(b_2 - b_3)]\xi^2 + a_1\rho_o\}\omega^2$$
$$+ \{[\mu\,(b_2 - b_3) - \frac{1}{4}\,\beta_2^2]\xi^2 + \mu a_1\}\xi^2 = 0\,. \tag{11.9}$$

Equation (11.9) governs the propagation of transverse waves in the isotropic medium. From (11.9) it is clear that for a given wavenumber ξ there are two ω^2, and solving for ω^2 and expanding for small ξ, we obtain

$$\hat{\omega}_1^2 = \frac{\mu}{\rho_o}\,\xi^2 + O(\xi^4)$$

$$\hat{\omega}_2^2 = \frac{a_1}{r^{(1)}\rho_o} + \frac{(b_2 - b_3)}{r^{(1)}\rho_o}\,\xi^2 + O(\xi^4)\,. \tag{11.10}$$

Again, and for the same reasons, it is clear that both $\hat{\omega}_1$ and $\hat{\omega}_2$ are real for real ξ, and on an ω vs ξ diagram there are two branches, one emanating from $\omega = 0$, $\xi = 0$, with positive initial slope $\sqrt{\mu/\rho_o}$, and the other emanating from $\omega = \sqrt{a_1/r^{(1)}\rho_o}$, $\xi = 0$, with zero initial slope and positive curvature $(b_2 - b_3)/\sqrt{r^{(1)}\rho_o a_1}$.

12. Dynamic Potentials

In the classical theory of isotropic linear elasticity it is possible to reduce the displacement equations of motion to wave equations in the Lamé potentials by means of the Helmholtz resolution. In this section the analogous

reduction is obtained for Eqs. (10.5) and (10.6), which were derived for an isotropic two constituent composite material, and the completeness of the representation is established following Sternberg[45] with minor modifications. To this end we write Eqs. (10.5) and (10.6) in the invariant vector form

$$\nabla^2(\mu \underset{\sim}{u} + \frac{1}{2}\beta_2 \underset{\sim}{w}) + \underset{\sim}{\nabla}\underset{\sim}{\nabla}\cdot[(\lambda+\mu)\underset{\sim}{u} + (\beta_1 + \frac{1}{2}\beta_2)\underset{\sim}{w}] = \rho\ddot{\underset{\sim}{u}}, \tag{12.1}$$

$$\nabla^2\left[\frac{1}{2}\beta_2\underset{\sim}{u} + (b_2 - b_3)\underset{\sim}{w}\right] + \underset{\sim}{\nabla}\underset{\sim}{\nabla}\cdot\left[\left(\beta_1 + \frac{1}{2}\beta_2\right)\underset{\sim}{u} + (b_1 + b_2 + b_3)\underset{\sim}{w}\right] - a_1\underset{\sim}{w} = r\rho_o\ddot{\underset{\sim}{w}}, \tag{12.2}$$

where we have taken the liberty of omitting the superscript (1) from $\underset{\sim}{w}$ and r. The substitution of the Helmholtz resolutions

$$\underset{\sim}{u} = \underset{\sim}{\nabla}\theta_1 + \underset{\sim}{\nabla}\times\underset{\sim}{H}_1,\quad \underset{\sim}{\nabla}\cdot\underset{\sim}{H}_1 = 0,$$

$$\underset{\sim}{w} = \underset{\sim}{\nabla}\chi_1 + \underset{\sim}{\nabla}\times\underset{\sim}{G}_1,\quad \underset{\sim}{\nabla}\cdot\underset{\sim}{G}_1 = 0, \tag{12.3}$$

into (12.1) and (12.2) yields

$$\underset{\sim}{\nabla}[(\lambda+2\mu)\nabla^2\theta_1 + (\beta_1+\beta_2)\nabla^2\chi_1 - \rho_o\ddot{\theta}_1] + \underset{\sim}{\nabla}\times\left[\mu\nabla^2\underset{\sim}{H}_1 + \frac{1}{2}\beta_2\nabla^2\underset{\sim}{G}_1 - \rho_o\ddot{\underset{\sim}{H}}_1\right] = 0, \tag{12.4}$$

$$\underset{\sim}{\nabla}[(\beta_1+\beta_2)\nabla^2\theta_1 + (b_1+2b_2)\nabla^2\chi_1 - a_1\chi_1 - r\rho_o\ddot{\chi}_1] + \underset{\sim}{\nabla}\times\left[\frac{1}{2}\beta_2\nabla^2\underset{\sim}{H}_1 + (b_2-b_3)\nabla^2\underset{\sim}{G}_1 - a_1\underset{\sim}{G}_1 - r\rho_o\ddot{\underset{\sim}{G}}_1\right] = 0. \tag{12.5}$$

Taking the divergence and the curl, respectively, of both (12.4) and (12.5), we obtain

$$\nabla^2[(\lambda+2\mu)\nabla^2\theta_1 + (\beta_1+\beta_2)\nabla^2\chi_1 - \rho_o\ddot{\theta}_1] = 0,\quad \nabla^2[\mu\nabla^2\underset{\sim}{H}_1 + \frac{1}{2}\beta_2\nabla^2\underset{\sim}{G}_1 - \rho_o\ddot{\underset{\sim}{H}}_1] = 0,$$

$$\nabla^2[(\beta_1+\beta_2)\nabla^2\theta_1 + (b_1+2b_2)\nabla^2\chi_1 - a_1\chi_1 - r\rho_o\ddot{\chi}_1] = 0,$$

$$\nabla^2\left[\frac{1}{2}\beta_2\nabla^2 H_1 + (b_2-b_3)\nabla^2\underset{\sim}{G}_1 - a_1\underset{\sim}{G}_1 - r\rho_o\ddot{\underset{\sim}{G}}_1\right] = 0, \tag{12.6}$$

where we have employed $(12.3)_{2,4}$ along with the identity

$$\underset{\sim}{\nabla}\times\underset{\sim}{\nabla}\times\underset{\sim}{v} = \underset{\sim}{\nabla}\underset{\sim}{\nabla}\cdot\underset{\sim}{v} - \nabla^2\underset{\sim}{v}, \tag{12.7}$$

in arriving at (12.6). Clearly, from (12.6), we may write

$$(\lambda + 2\mu)\nabla^2\theta_1 + (\beta_1 + \beta_2)\nabla^2\chi_1 - \rho_o\ddot{\theta}_1 = a\,, \quad \mu\nabla^2 H_1 + \frac{1}{2}\beta_2\nabla^2 \underset{\sim}{G}_1 - \rho_o\ddot{\underset{\sim}{H}}_1 = b\,,$$

$$(\beta_1 + \beta_2)\nabla^2\theta_1 + (b_1 + 2b_2)\nabla^2\chi_1 - a_1\chi_1 - r\rho_o\ddot{\chi}_1 = d\,,$$

$$\frac{1}{2}\beta_2\nabla^2\underset{\sim}{H}_1 + (b_2 - b_3)\nabla^2\underset{\sim}{G}_1 - a_1\underset{\sim}{G}_1 - r\rho_o\ddot{\underset{\sim}{G}}_1 = \underset{\sim}{e}\,, \tag{12.8}$$

where

$$\nabla^2 a = 0\,, \quad \nabla^2\underset{\sim}{b} = 0\,, \quad \underset{\sim}{\nabla}\cdot\underset{\sim}{b} = 0\,, \quad \nabla^2 d = 0\,, \quad \nabla^2\underset{\sim}{e} = 0\,, \quad \nabla\cdot\underset{\sim}{e} = 0\,. \tag{12.9}$$

Now, let

$$\theta = \theta_1 + A\,, \quad \underset{\sim}{H} = \underset{\sim}{H}_1 + \underset{\sim}{B}\,, \quad \chi = \chi_1 + D\,, \quad \underset{\sim}{G} = \underset{\sim}{G}_1 + \underset{\sim}{E}\,, \tag{12.10}$$

where A, $\underset{\sim}{B}$, D and $\underset{\sim}{E}$ are particular functions to be selected in order that

$$\rho_o\ddot{A} = a\,, \quad \rho_o\ddot{\underset{\sim}{B}} = \underset{\sim}{b}\,, \quad a_1 D + r\rho_o\ddot{D} = d\,, \quad a_1\underset{\sim}{E} + r\rho_o\ddot{\underset{\sim}{E}} = \underset{\sim}{e}\,. \tag{12.11}$$

To this end we take A, $\underset{\sim}{B}$, D and $\underset{\sim}{E}$ in the forms

$$A = \frac{1}{\rho_o}\int_0^t\int_0^\tau a(\underset{\sim}{X},s)\,ds\,d\tau\,, \quad \underset{\sim}{B} = \frac{1}{\rho_o}\int_0^t\int_0^\tau \underset{\sim}{b}(\underset{\sim}{X},s)\,ds\,d\tau\,,$$

$$D = \frac{1}{\omega_o r\rho_o}\int_0^t d(\underset{\sim}{X},s)\sin\omega_o(t-s)\,ds\,, \quad \underset{\sim}{E} = \frac{1}{\omega_o r\rho_o}\int_0^t \underset{\sim}{e}(\underset{\sim}{X},s)\sin\omega_o(t-s)\,ds\,, \tag{12.12}$$

where

$$\omega_o^2 = a_1/r\rho_o\,, \tag{12.13}$$

and on account of (12.9)

$$\nabla^2 A = 0\,, \quad \nabla^2\underset{\sim}{B} = 0\,, \quad \underset{\sim}{\nabla}\cdot\underset{\sim}{B} = 0\,, \quad \nabla^2 D = 0\,, \quad \nabla^2\underset{\sim}{E} = 0\,, \quad \underset{\sim}{\nabla}\cdot\underset{\sim}{E} = 0\,. \tag{12.14}$$

Moreover, from $(12.9)_{4-5}$ and $(12.11)_{3-4}$ we have

$$\nabla^2\ddot{D} = 0\,, \quad \nabla^2\ddot{\underset{\sim}{E}} = 0\,. \tag{12.15}$$

Substituting from (12.10) into (12.8) and employing (12.11) and (12.14), we obtain

$$(\lambda + 2\mu)\nabla^2\theta + (\beta_1 + \beta_2)\nabla^2\chi - \rho_o\ddot{\theta} = 0\,, \quad \mu\nabla^2\underset{\sim}{H} + \frac{1}{2}\,\beta_2\nabla^2\underset{\sim}{G} - \rho_o\ddot{\underset{\sim}{H}} = 0\,,$$

$$(\beta_1 + \beta_2)\nabla^2\theta + (b_1 + 2b_2)\nabla^2\chi - a_1\chi - r\rho_o\ddot{\chi} = 0\,,$$

$$\frac{1}{2}\,\beta_2\nabla^2\underset{\sim}{H} + (b_2 - b_3)\nabla^2\underset{\sim}{G} - a_1\underset{\sim}{G} - r\rho_o\ddot{\underset{\sim}{G}} = 0\,. \tag{12.16}$$

From (12.3) and (12.10) we may write

$$\underset{\sim}{u} = \underset{\sim}{\nabla}\theta + \underset{\sim}{\nabla}\times\underset{\sim}{H} + \hat{\underset{\sim}{u}}\,, \quad \underset{\sim}{w} = \underset{\sim}{\nabla}\chi + \underset{\sim}{\nabla}\times\underset{\sim}{G} + \hat{\underset{\sim}{w}}\,, \tag{12.17}$$

where

$$\hat{\underset{\sim}{u}} = -\,\underset{\sim}{\nabla}A - \underset{\sim}{\nabla}\times\underset{\sim}{B}\,, \quad \hat{\underset{\sim}{w}} = -\,\underset{\sim}{\nabla}D - \underset{\sim}{\nabla}\times\underset{\sim}{E}\,. \tag{12.18}$$

Equations (12.18) and (12.14), along with the identity (12.7) applied to $\underset{\sim}{B}$ and $\underset{\sim}{E}$, imply that

$$\underset{\sim}{\nabla}\cdot\hat{\underset{\sim}{u}} = 0\,, \quad \underset{\sim}{\nabla}\times\hat{\underset{\sim}{u}} = 0\,, \quad \underset{\sim}{\nabla}\cdot\hat{\underset{\sim}{w}} = 0\,, \quad \underset{\sim}{\nabla}\times\hat{\underset{\sim}{w}} = 0\,. \tag{12.19}$$

Therefore, there exist functions $\zeta(\underset{\sim}{X},t)$ and $\eta(\underset{\sim}{X},t)$ such that

$$\hat{\underset{\sim}{u}} = \underset{\sim}{\nabla}\zeta\,, \quad \nabla^2\zeta = 0\,, \quad \hat{\underset{\sim}{w}} = \underset{\sim}{\nabla}\eta\,, \quad \nabla^2\eta = 0\,, \tag{12.20}$$

and (12.17) and (12.20) permit us to write

$$\underset{\sim}{u} = \underset{\sim}{\nabla}\theta + \underset{\sim}{\nabla}\zeta + \underset{\sim}{\nabla}\times\underset{\sim}{H}\,, \quad \underset{\sim}{w} = \underset{\sim}{\nabla}\chi + \underset{\sim}{\nabla}\eta + \underset{\sim}{\nabla}\times\underset{\sim}{G}\,. \tag{12.21}$$

Substituting from (12.21) into (12.1) and (12.2) and employing (12.16) and (12.20), we obtain

$$\underset{\sim}{\nabla}\ddot{\zeta} = 0\,, \quad \underset{\sim}{\nabla}(a_1\eta + r\rho_o\ddot{\eta}) = 0\,. \tag{12.22}$$

Hence, ζ and η must have the respective forms

$$\zeta = \alpha_1(t) + t\beta_1(\underset{\sim}{X}) + \gamma_1(\underset{\sim}{X})\,, \quad \eta = \alpha_2(t) + \beta_2(\underset{\sim}{X})\cos\omega_o t + \gamma_2(\underset{\sim}{X})\sin\omega_o t\,. \tag{12.23}$$

Equations $(12.20)_{2,4}$ and (12.23) indicate that

$$\nabla^2\beta_1 = 0\,, \quad \nabla^2\gamma_1 = 0\,, \quad \nabla^2\beta_2 = 0\,, \quad \nabla^2\gamma_2 = 0\,. \tag{12.24}$$

Finally, let us define

$$\varphi = \theta + \zeta - \alpha_1, \quad \psi = \chi + \eta - \alpha_2, \tag{12.25}$$

which, with (12.21), $(12.3)_{2,4}$, $(12.10)_{2,4}$, and $(12.14)_{3,6}$ enables us to write

$$\begin{aligned} \underset{\sim}{u} &= \underset{\sim}{\nabla}\varphi + \underset{\sim}{\nabla} \times \underset{\sim}{H}, \quad \underset{\sim}{\nabla} \cdot \underset{\sim}{H} = 0, \\ \underset{\sim}{w} &= \underset{\sim}{\nabla}\psi + \underset{\sim}{\nabla} \times \underset{\sim}{G}, \quad \underset{\sim}{\nabla} \cdot \underset{\sim}{G} = 0, \end{aligned} \tag{12.26}$$

and from (12.16) and (12.23) - (12.25) we see that the potentials satisfy

$$\begin{aligned} (\lambda + 2\mu)\nabla^2\varphi + (\beta_1 + \beta_2)\nabla^2\psi - \rho_o\ddot{\varphi} = 0, \quad \mu\nabla^2\underset{\sim}{H} + \frac{1}{2}\beta_2\nabla^2\underset{\sim}{G} - \rho_o\ddot{\underset{\sim}{H}} = 0, \\ (\beta_1 + \beta_2)\nabla^2\varphi + (b_1 + 2b_2)\nabla^2\psi - a_1\psi - r\rho_o\ddot{\psi} = 0, \\ \frac{1}{2}\beta_2\nabla^2\underset{\sim}{H} + (b_2 - b_3)\nabla^2\underset{\sim}{G} - a_1\underset{\sim}{G} - r\rho_o\ddot{\underset{\sim}{G}} = 0, \end{aligned} \tag{12.27}$$

and we have shown that the representation is complete. It should be noted that $\underset{\sim}{u}(\underset{\sim}{X},t)$ and $\underset{\sim}{w}(\underset{\sim}{X},t)$, given by $(12.26)_{1,3}$, satisfy (12.1) and (12.2) provided (12.27) hold even if $\underset{\sim}{H}$ and $\underset{\sim}{G}$ are not solenoidal, as may be confirmed by direct substitution.

13. Static Potentials

A complete solution of the displacement equations of equilibrium for an isotropic two-constituent composite material

$$\nabla^2\left(\mu\underset{\sim}{u} + \frac{1}{2}\beta_2\underset{\sim}{w}\right) + \underset{\sim}{\nabla}\underset{\sim}{\nabla} \cdot \left[(\lambda + \mu)\underset{\sim}{u} + \left(\beta_1 + \frac{1}{2}\beta_2\right)\underset{\sim}{w}\right] + \rho_o\underset{\sim}{f} = 0, \tag{13.1}$$

$$\begin{aligned} \nabla^2\left[\frac{1}{2}\beta_2\underset{\sim}{u} + (b_2 - b_3)\underset{\sim}{w}\right] + \underset{\sim}{\nabla}\underset{\sim}{\nabla} \cdot \left[\left(\beta_1 + \frac{1}{2}\beta_2\right)\underset{\sim}{u} + (b_1 + b_2 + b_3)\underset{\sim}{w}\right] \\ - a_1\underset{\sim}{w} + [r/(1 + r)]\,\rho_o\tilde{\underset{\sim}{f}}^{(1)} = 0, \end{aligned} \tag{13.2}$$

is obtained in terms of stress functions, which reduce to the Papkovitch functions of classical elasticity. The procedure follows that of Mindlin[46,47].

The substitution of the Helmholtz resolutions

$$\underset{\sim}{u} = \underset{\sim}{\nabla}\varphi + \nabla \times \underset{\sim}{H}, \quad \underset{\sim}{\nabla} \cdot \underset{\sim}{H} = 0,$$

$$\underset{\sim}{w} = \underset{\sim}{\nabla}\psi + \underset{\sim}{\nabla} \times \underset{\sim}{V}, \quad \underset{\sim}{\nabla} \cdot \underset{\sim}{V} = 0, \tag{13.3}$$

into (13.1) and (13.2) yields

$$\nabla^2[\alpha_1 \underset{\sim}{\nabla}\varphi + \underset{\sim}{\nabla} \times \underset{\sim}{H} + \alpha_2 \underset{\sim}{\nabla}\psi + \alpha_3 \underset{\sim}{\nabla} \times \underset{\sim}{V}] + \hat{\underset{\sim}{f}} = 0, \tag{13.4}$$

$$\underset{\sim}{\nabla}[(\beta_1 + \beta_2)\nabla^2\varphi + (b_1 + 2b_2)\nabla^2\psi - a_1\psi] +$$

$$\underset{\sim}{\nabla} \times \left[\frac{1}{2}\beta_2\nabla^2\underset{\sim}{H} + (b_2 - b_3)\nabla^2\underset{\sim}{V} - a_1\underset{\sim}{V}\right] + \hat{\underset{\sim}{f}}^{(1)} = 0, \tag{13.5}$$

where for convenience we have introduced the definitions

$$\hat{\underset{\sim}{f}} = \rho_o \underset{\sim}{f}, \quad \hat{\underset{\sim}{f}}^{(1)} = [r/(1+r)]\rho_o \tilde{\underset{\sim}{f}}^{(1)}, \tag{13.6}$$

$$\alpha_1 = \frac{\lambda + 2\mu}{\mu}, \quad \alpha_2 = \frac{\beta_1 + \beta_2}{\mu}, \quad \alpha_3 = \frac{\beta_2}{2\mu}. \tag{13.7}$$

Let us define a vector function $\underset{\sim}{B}$ by

$$\underset{\sim}{B} = \alpha_1 \underset{\sim}{\nabla}\varphi + \underset{\sim}{\nabla} \times \underset{\sim}{H} + \alpha_2 \underset{\sim}{\nabla}\psi + \alpha_3 \underset{\sim}{\nabla} \times \underset{\sim}{V}, \tag{13.8}$$

then

$$\nabla^2 \underset{\sim}{B} = -\hat{\underset{\sim}{f}}/\mu. \tag{13.9}$$

Taking the divergence of (13.8), we obtain

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = \nabla^2(\alpha_1\varphi + \alpha_2\psi), \tag{13.10}$$

which, with the definition

$$\chi = \alpha_1\varphi + \alpha_2\psi, \tag{13.11}$$

enables us to write

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = \nabla^2\chi. \tag{13.12}$$

Since

$$\nabla^2(\underset{\sim}{r} \cdot \underset{\sim}{B}) = 2\underset{\sim}{\nabla} \cdot \underset{\sim}{B} + \underset{\sim}{r} \cdot \nabla^2 \underset{\sim}{B}, \tag{13.13}$$

from (13.12) and (13.9) we have

$$\nabla^2 B_o = \underset{\sim}{r} \cdot \hat{\underset{\sim}{f}}/\mu, \tag{13.14}$$

where we have employed the definition

$$B_o = 2\chi - \underset{\sim}{r} \cdot \underset{\sim}{B}. \tag{13.15}$$

It should be noted that Eqs. (13.9) and (13.14) are exactly the same as in classical linear elasticity.

We must now eliminate $\underset{\sim}{H}$ and φ from the representation in order to express Eq. (13.5) in terms of $\underset{\sim}{B}$, B_o, ψ and $\underset{\sim}{V}$. To this end we first take the curl of (13.8) and employ $(13.3)_{2,4}$ to obtain

$$\nabla^2 \underset{\sim}{H} = - \underset{\sim}{\nabla} \times \underset{\sim}{B} - \alpha_3 \nabla^2 \underset{\sim}{V}. \tag{13.16}$$

We now substitute from (13.15) into (13.11) to obtain

$$\varphi = [(\underset{\sim}{r} \cdot \underset{\sim}{B} + B_o)/2\alpha_1] - (\alpha_2/\alpha_1)\psi. \tag{13.17}$$

Taking the divergence and the curl, respectively, of (13.5) and employing $(13.3)_{2,4}$, we have

$$\nabla^2[(\beta_1 + \beta_2)\nabla^2\varphi + (b_1 + 2b_2)\nabla^2\psi - a_1\psi] + \underset{\sim}{\nabla} \cdot \hat{\underset{\sim}{f}}^{(1)} = 0, \tag{13.18}$$

$$\nabla^2\left[\frac{1}{2}\beta_2\nabla^2\underset{\sim}{H} + (b_2 - b_3)\nabla^2\underset{\sim}{V} - a_1\underset{\sim}{V}\right] - \underset{\sim}{\nabla} \times \hat{\underset{\sim}{f}}^{(1)} = 0. \tag{13.19}$$

Now, substituting from (13.16) and (13.17) into (13.18) and (13.19) and rearranging terms, we find

$$\nabla^2(1 - \ell_1^2\nabla^2)\psi = \kappa_1\nabla^4(\underset{\sim}{r} \cdot \underset{\sim}{B} + B_o) + \underset{\sim}{\nabla} \cdot \hat{\underset{\sim}{f}}^{(1)}/a_1, \tag{13.20}$$

$$\nabla^2(1 - \ell_2^2\nabla^2)\underset{\sim}{V} = -\kappa_2\nabla^2\underset{\sim}{\nabla} \times \underset{\sim}{B} - \underset{\sim}{\nabla} \times \hat{\underset{\sim}{f}}^{(1)}/a_1, \tag{13.21}$$

where

$$\ell_1^2 = \frac{b_1 + 2b_2}{a_1} - \frac{(\beta_1+\beta_2)^2}{a_1(\lambda+2\mu)}, \quad \varkappa_1 = \frac{(\beta_1+\beta_2)\mu}{2(\lambda+2\mu)a_1},$$

$$\ell_2^2 = \frac{b_2 - b_3}{a_1} - \frac{\beta_2^2}{4\mu a_1}, \quad \varkappa_2 = \frac{\beta_2}{2a_1}. \tag{13.22}$$

The substitution of (13.8) and the gradient of (13.17) into $(13.3)_1$ yields

$$\underset{\sim}{u} = \underset{\sim}{B} - \alpha_1' \underset{\sim}{\nabla}(\underset{\sim}{r}\cdot\underset{\sim}{B} + B_o) - \frac{(\beta_1+\beta_2)}{\lambda+2\mu}\underset{\sim}{\nabla}\psi - \frac{\beta_2}{2\mu}\underset{\sim}{\nabla}\times\underset{\sim}{V}, \tag{13.23}$$

where

$$\alpha_1' = (\lambda+\mu)/2(\lambda+2\mu). \tag{13.24}$$

Thus, finally the representation consists of the differential equations (13.9), (13.14), (13.20) and (13.21) along with the expressions (13.23) and $(13.3)_3$. The functions $\underset{\sim}{B}$ and B_o reduce to Papkovitch functions when β_1 and β_2 vanish.

14. Concentrated Forces

In this section we consider first the concentrated force and then the concentrated relative force located at the origin in an infinite isotropic two-constituent composite medium. In an infinite medium acted upon statically by body forces $\hat{\underset{\sim}{f}}$ and relative body forces $\hat{\underset{\sim}{f}}^{(1)}$ we have Eqs. (13.9), (13.14), (13.20), $(13.3)_4$ and (13.21) for the potential functions $\underset{\sim}{B}$, B_o, ψ and $\underset{\sim}{V}$, which enable the determination of the displacement fields $\underset{\sim}{u}$ and $\underset{\sim}{w}$ through (13.23) and $(13.3)_3$.

In the case of the concentrated force we have $\hat{\underset{\sim}{f}}^{(1)} = 0$ everywhere and $\hat{\underset{\sim}{f}} = 0$ outside a region V' encompassing the origin and containing a nonvanishing field of parallel forces $\hat{\underset{\sim}{f}}$. A concentrated force is defined in the usual way by

$$\underset{\sim}{P} = \lim_{V'\to 0} \int_V \hat{\underset{\sim}{f}}\, dV. \tag{14.1}$$

For the case under consideration Eqs. (13.9), (13.14) and $(13.3)_4$ take the form shown and Eqs. (13.20) and (13.21) may be integrated twice to give

$$(1 - \ell_1^2\nabla^2)\psi = \varkappa_1\nabla^2(\underset{\sim}{r}\cdot\underset{\sim}{B}+B_o), \tag{14.2}$$

$$(1 - \ell_2^2\nabla^2)\underset{\sim}{V} = -\varkappa_2\underset{\sim}{\nabla}\times\underset{\sim}{B}. \tag{14.3}$$

Since, for the infinite medium, solutions of equations having the respective forms

$$\nabla^2\theta = \sigma, \quad (1 - \ell^2\nabla^2)\tilde{\theta} = \tilde{\sigma}, \tag{14.4}$$

can be written in the respective forms[48]

$$\theta = -\frac{1}{4\pi}\int_{V_Q}\frac{\sigma(Q)}{V_1}\,dV_Q, \quad \tilde{\theta} = \frac{1}{4\pi\ell^2}\int_{V_Q}\frac{e^{-r_1/\ell}}{r_1}\tilde{\sigma}(Q)\,dV_Q, \tag{14.5}$$

where

$$r_1 = [(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{\frac{1}{2}}, \tag{14.6}$$

is the distance from the field point P at $\underset{\sim}{R}$ to the source point Q at $\underset{\sim}{r}'$ and

$$\underset{\sim}{R}\cdot\underset{\sim}{R} = x^2+y^2+z^2, \quad \underset{\sim}{r}'\cdot\underset{\sim}{r}' = \xi^2+\eta^2+\zeta^2, \tag{14.7}$$

we have from (13.9), (13.14), (13.20) and (13.21)

$$\underset{\sim}{B} = \frac{1}{4\pi\mu}\int_V\frac{\hat{\underset{\sim}{f}}\,dV}{r_1}, \quad B_o = -\frac{1}{4\pi\mu}\int_V\frac{\underset{\sim}{r}'\cdot\hat{\underset{\sim}{f}}\,dV}{r_1}, \tag{14.8}$$

$$\psi = \frac{\varkappa_1}{4\pi\ell_1^2}\int_V\frac{e^{-r_1/\ell_1}}{r_1}\nabla_Q^2(\underset{\sim}{r}'\cdot\underset{\sim}{B}+B_o)\,dV, \tag{14.9}$$

$$\underset{\sim}{V} = -\frac{\varkappa_2}{4\pi\ell_2^2}\int_V\frac{e^{-r_1/\ell_2}}{r_1}\underset{\sim}{\nabla}_Q\times\underset{\sim}{B}\,dV. \tag{14.10}$$

Then, from (14.1) and (14.8) in the usual way, we obtain

$$\underset{\sim}{B} = \underset{\sim}{P}/4\pi\mu R, \quad B_o = 0, \tag{14.11}$$

since

$$\lim_{V'\to 0} r_1 = R, \quad \lim_{V'\to 0} \underset{\sim}{r}' = 0. \tag{14.12}$$

The relations in (14.11) naturally are the same as in the classical theory of isotropic linear elasticity. Since

$$\underset{\sim}{P} = P\underset{\sim}{e}_z , \tag{14.13}$$

where $\underset{\sim}{e}_z$ denotes the unit base vector in the z-direction, we have

$$\underset{\sim}{R} \cdot \underset{\sim}{B} = zP/4\pi\mu \sqrt{\hat{r}^2 + z^2} , \quad \underset{\sim}{\nabla} \times \underset{\sim}{B} = (P/4\pi\mu R^2)\underset{\sim}{e}_\theta , \tag{14.14}$$

where $\underset{\sim}{e}_\theta$ denotes the unit base vector in the cylindrical coordinate θ-direction, $\hat{r}$ is the magnitude of the cylindrical coordinate radial position and, of course, we have

$$\hat{r}^2 + z^2 = R^2 . \tag{14.15}$$

Substituting from $(14.14)_1$ into (14.9) and converting to cylindrical coordinates, we have

$$\psi = -\frac{\varkappa_1 P}{8\pi^2 \mu \ell_1^2} \int_0^\infty \int_0^{2\pi} \int_{-\infty}^{\infty} \frac{e^{-r_1/\ell_1}\,\zeta}{r_1(\zeta^2 + \hat{r}'^2)^{\frac{3}{2}}} \, r' dr' d\theta' d\zeta \tag{14.16}$$

where in cylindrical coordinates

$$\hat{r}_1 = [\hat{r}^2 + \hat{r}'^2 - 2\hat{r}\hat{r}' \cos(\theta - \theta') + (z - \zeta)^2]^{\frac{1}{2}} , \tag{14.17}$$

and

$$\hat{r}^2 = x^2 + y^2 , \quad \hat{r}'^2 = \xi^2 + \eta^2 , \quad \theta = \tan^{-1} x/y , \quad \theta' = \tan^{-1} \xi/\eta . \tag{14.18}$$

At this point it should be noted that since the integral over θ' in (14.16) has an interval of 2π which is the period of $\cos(\theta - \theta')$ in (14.17), the resulting expression for ψ in (14.16) is actually independent of θ.

Since $\underset{\sim}{\nabla} \times \underset{\sim}{B}$ is spherically symmetric, it is advantageous to return to (14.13) rather than to use (14.10) directly to obtain $\underset{\sim}{V}$ because an ordinary differential equation in R results. Substituting from $(14.14)_2$ into (14.3), we obtain

$$(1 - \ell_2^2\nabla^2)\underset{\sim}{V} = (\varkappa_2 P/4\pi\mu R^2) e_\theta \, . \tag{14.19}$$

As the solution of (14.19) we can take

$$\underset{\sim}{V} = V_\theta(R)\underset{\sim}{e}_\theta \, , \tag{14.20}$$

because

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{V} = 0 \, , \quad \partial\underset{\sim}{e}_\theta/\partial R = 0 \, . \tag{14.21}$$

Then ∇^2 takes the spherically symmetric form

$$\nabla^2 = R^{-2}\partial(R^2\partial/\partial R)/\partial R \, , \tag{14.22}$$

and substituting from (14.20) and (14.22) into (14.19) and employing $(14.21)_2$, we obtain

$$[1 - \ell_2^2 R^{-2} d(R^2 d/dR)/dR]\, V_\theta = \varkappa_2 P/4\pi\mu R^2 \, . \tag{14.23}$$

On account of the relation

$$R^{-1}\, d^2(RV_\theta)/dR^2 = R^{-2} d(R^2 dV_\theta/dR)/dR \, , \tag{14.24}$$

Eq. (14.23) can be written

$$RV_\theta - \ell_2^2 d^2(RV_\theta)/dR^2 = \varkappa_2 P/4\pi\mu R \, , \tag{14.25}$$

the inhomogeneous solution of which is

$$V_\theta = - \frac{\varkappa_2 P}{4\pi\mu\ell_2 R} \int_0^R \frac{1}{s} \sinh \frac{(R-s)}{\ell_2}\, ds \, . \tag{14.26}$$

Thus, ψ and $\underset{\sim}{V}$ for the concentrated body force have been written as definite integrals, and we carry the solution for the concentrated force no further. Clearly, if the coupling coefficients β_1 and β_2 vanish, the solution reduces to that of the classical theory of elasticity.

For the case of small ℓ_1 and ℓ_2, which should be most common, asymptotic representations of the solution for ψ and V_θ in terms of simple functions for $R \ll \ell_1, \ell_2$ and $R \gg \ell_1, \ell_2$ can readily be obtained. These asymptotic solutions can be matched in the intermediate region. However, although V_θ can be matched relatively easily since it satisfies an ordinary differential equation, the matching of ψ requires some effort because it satisfies a partial differential equation. Consequently, these asymptotic representations will not be treated here.

In the case of the concentrated relative force we have $\hat{\underset{\sim}{f}} = 0$ everywhere and $\hat{\underset{\sim}{f}}^{(1)} = 0$ outside a region V' enclosing the origin and containing a non-vanishing field of parallel relative forces $\hat{\underset{\sim}{f}}^{(1)}$. A concentrated relative force is defined by

$$\underset{\sim}{\mathcal{G}} = \lim_{V' \to 0} \int_{V'} \underset{\sim}{f}^{(1)}\, dV \,. \tag{14.27}$$

For this case Eqs. (13.9) and (13.14) take the form shown and by virtue of (14.8), we have

$$\underset{\sim}{B} = 0\,, \quad B_o = 0\,. \tag{14.28}$$

In addition, from Eqs. (13.20) and (13.21), we have

$$\nabla^2 (1 - \ell_1^2 \nabla^2)\psi = \underset{\sim}{\nabla} \cdot \hat{\underset{\sim}{f}}^{(1)}/a_1 \,, \tag{14.29}$$

$$\nabla^2 (1 - \ell_2^2 \nabla^2)\underset{\sim}{V} = - \underset{\sim}{\nabla} \times \hat{f}^{(1)}/a_1 \,. \tag{14.30}$$

Since for the infinite medium solutions of the equation having the form

$$\nabla^2 (1 - \ell^2 \nabla^2)\theta = \sigma \,, \tag{14.31}$$

can be written in the form[48]

$$\theta = - \frac{1}{4\pi} \int_V r_1^{-1} (1 - e^{-r_1/\ell})\sigma \, dV \,, \tag{14.32}$$

we have, from (14.29) and (14.30),

$$\psi = -\frac{1}{4\pi a_1}\int_S \underset{\sim}{n}\cdot[r_1^{-1}(1-e^{-r_1/\ell_1})\hat{\underset{\sim}{f}}^{(1)}]dS + \frac{1}{4\pi a_1}\int_V \hat{\underset{\sim}{f}}^{(1)}\cdot\underset{\sim}{\nabla}[r_1^{-1}(1-e^{-r_1/\ell_1})]dV, \tag{14.33}$$

$$\underset{\sim}{V} = \frac{1}{4\pi a_1}\int_S \underset{\sim}{n}\times[r_1^{-1}(1-e^{-r_1/\ell_2})\hat{\underset{\sim}{f}}^{(1)}]dS + \frac{1}{4\pi a_1}\int_V \hat{\underset{\sim}{f}}^{(1)}\times\underset{\sim}{\nabla}[r_1^{-1}(1-e^{-r_1/\ell_2})]dV, \tag{14.34}$$

and we note that the surface integrals in (14.33) and (14.34) vanish because $\hat{\underset{\sim}{f}}^{(1)} = 0$ outside V'. By virtue of (14.12) and (14.27), in the limit $V' \to 0$ Eqs. (14.33) and (14.34), respectively, reduce to

$$\psi = \frac{1}{4\pi a_1}\underset{\sim}{\mathcal{L}}\cdot\underset{\sim}{\nabla}\left[\frac{1}{R}\,(1-e^{-R/\ell_1})\right],\quad \underset{\sim}{V} = \frac{1}{4\pi a_1}\underset{\sim}{\mathcal{L}}\times\underset{\sim}{\nabla}\left[\frac{1}{R}\,(1-e^{-R/\ell_2})\right], \tag{14.35}$$

which, with (14.28), are the stress functions for the concentrated relative force.

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24. We are primarily interested in the non-dissipative case and include viscous dissipation only to indicate the changes in the basic description that occur due to viscous type dissipation. As a consequence, we consider the simplest (Kelvin) type of viscoelastic dissipation in solids.

25. Ref.14, Eqs.(21.1) and (21.2).

26. Ref.11, Secs.1.8 - 1.12.

27. Ref.12, Chap.III, Secs.1 and 2.

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32. Ref.23, Eq.(31.1).

33. This procedure was given the name "Principle of Equipresence" by Truesdell. See Ref.23, Sec.293.

34. Ref.14, Eq.(27.12).

35. Ref.14, Eq.(27.15).

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38. Since all invariants of vectors and second-rank tensors given in Table I of Ref.37 for the proper orthogonal group consist of traces of combinations of scalar products, it is a relatively easy matter to show that all the other dissipative rate-type invariants may be expressed in terms of the basic rate-type invariants R_{MN}, $Z^{(1)}_{MN}$, $B^{(1)}_{M}$, $Z^{(2)}_{MN}$ and $B^{(2)}_{M}$ along with the state-type invariant C_{LM} (or E_{LM}).

39. Ref.23, Secs.192 and 193.

40. Ref.23, Sec.20, Eq.(20.8).

41. Ref.23, Secs.17 and 18, Eq.(18.2).

42. The relation between U_N and the actual velocity u of the surface of discontinuity is given in P.J. Chen, "Growth and Decay of Waves in Solids," in Encyclopedia of Physics, edited by C. Truesdell (Springer-Verlag, Berlin, 1973), Vol.IVa/3, Sec.4.

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FIGURE CAPTIONS

Figure 1	Schematic Diagram Showing the Relative Displacements of the Interacting Continua
Figure 2	Schematic Diagram Showing the Linear Momentum and Force and Couple Vectors Acting in Continuum 1.
Figure 3	Schematic Diagram Showing the Linear Momentum and Force and Couple Vectors Acting in Continuum 2.
Figure 4	Schematic Diagram Showing the Linear Momentum and Force and Couple Vectors Acting in Continuum 3.

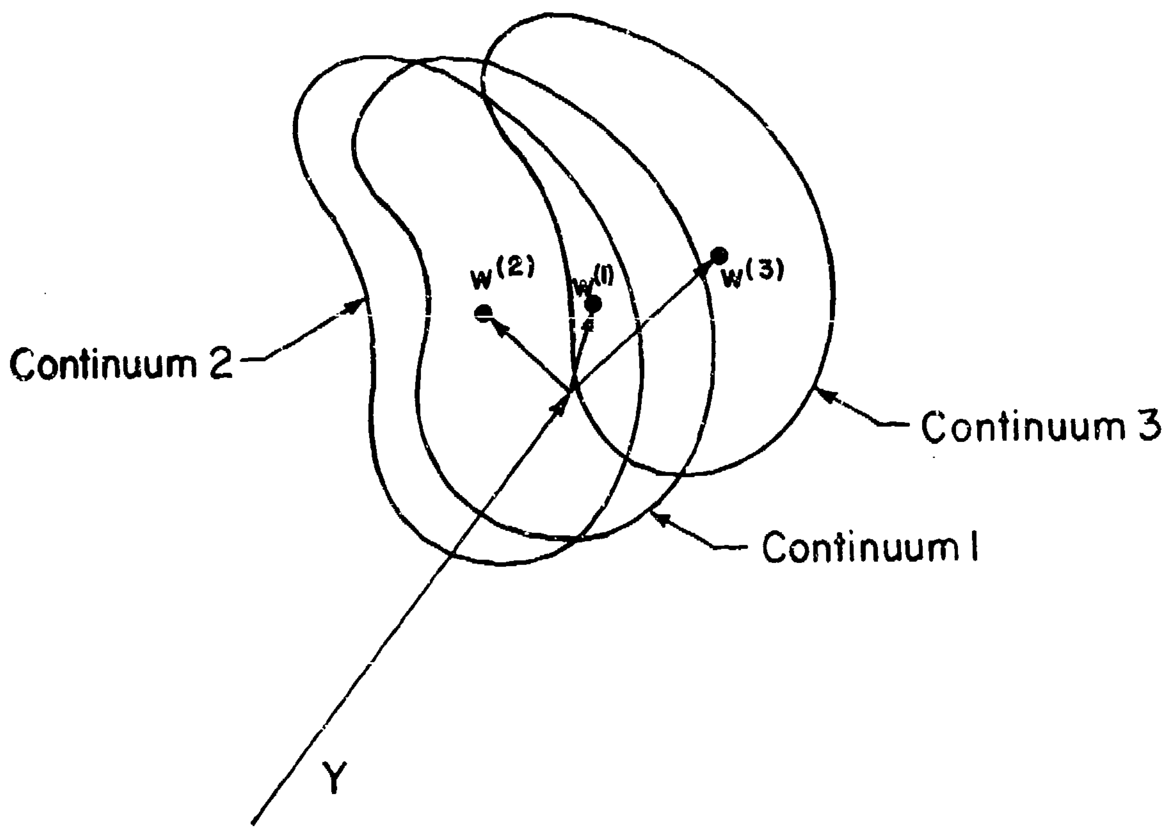
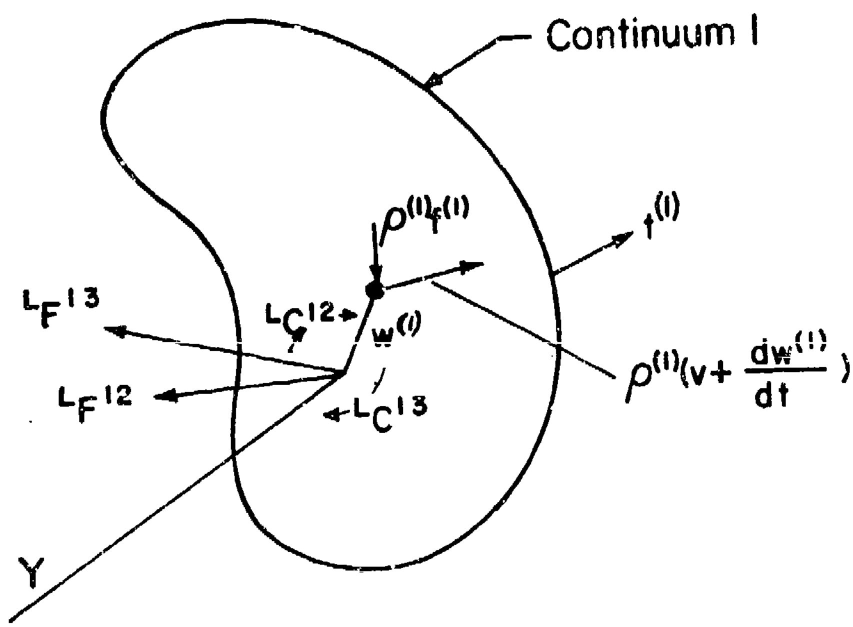


Figure 1



Continuum I
$\rho^{(I)}f^{(I)}$
$t^{(I)}$
$^{L}F^{13}$
$^{L}C^{12}$
$w^{(I)}$
$\rho^{(I)}(v+\frac{dw^{(I)}}{dt})$
$^{L}F^{12}$
$^{L}C^{13}$
Y

Figure 2

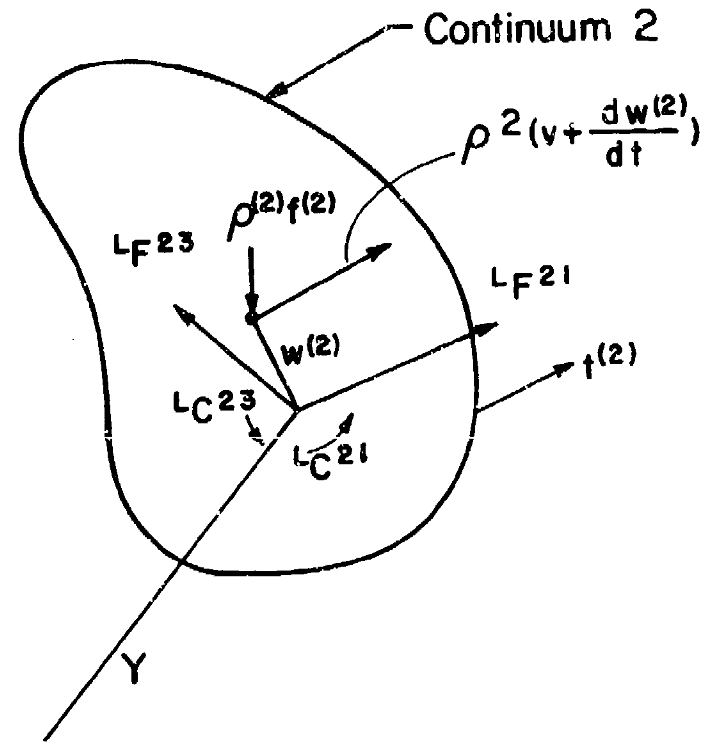

Continuum 2
$\rho^2(v + \frac{dw^{(2)}}{dt})$
$\rho^{(2)} f^{(2)}$
${}^L F^{23}$
${}^L F^{21}$
$w^{(2)}$
$t^{(2)}$
${}^L C^{23}$
${}^L C^{21}$
Y

Figure 3

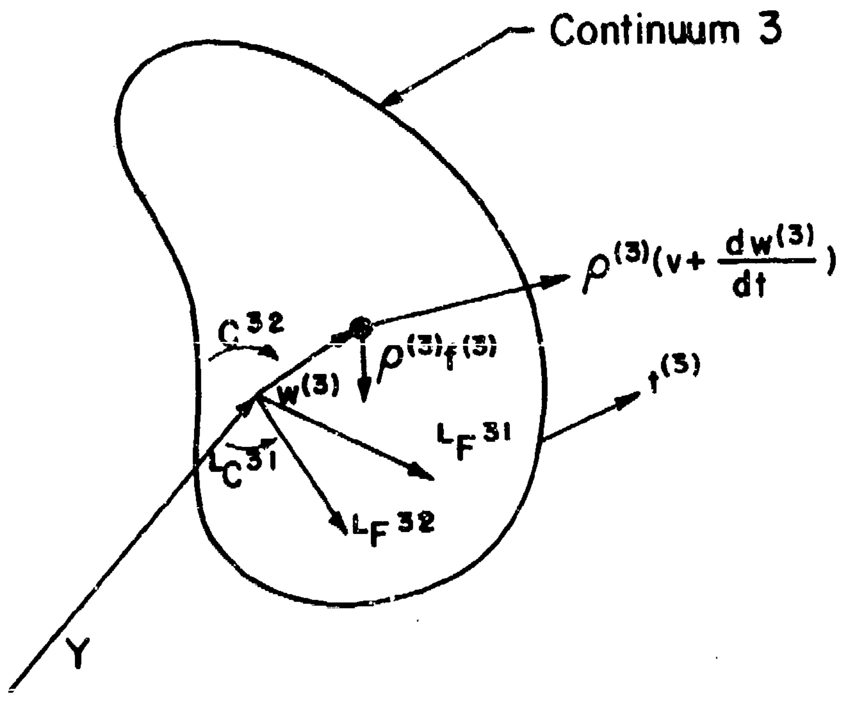

Continuum 3
$\rho^{(3)}(v + \frac{dw^{(3)}}{dt})$
C^{32}
$\rho^{(3)}f^{(3)}$
$w^{(3)}$
$t^{(3)}$
$^{L}F^{31}$
$^{L}C^{31}$
$^{L}F^{32}$
Y

Figure 4